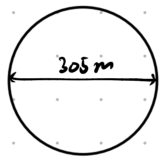
Lec 1 - 1/16

Syllabus overview (see bCourses)

OH : This week Fri 11am - 1pm, 403 Physics S

Start of Material

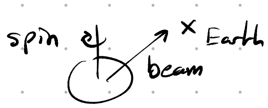
Puerto Rico, 1.5 hr W of San Jose : ARECIBO Radio Observatory



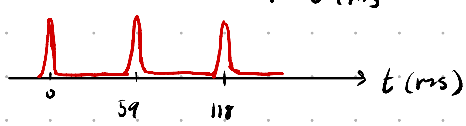
1974 Russell Hulse & Joseph Taylor

Radio signal from Pulsar 20,000 lyr

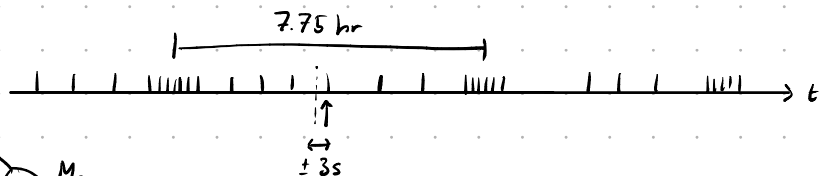
neutron-star

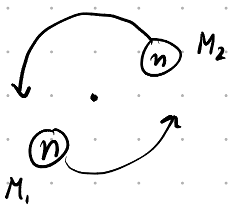


$T = 59\,ms$



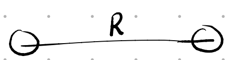
Noticed:





Period of orbit of binary system is $T = 7.75\,hr$

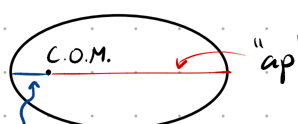
$M_1 \approx M_2 \simeq 1.4 M_\odot$, $M_\odot$ = mass of sun $\approx 2 \times 10^{30}\,kg$



$R_\odot$ = radius of sun $\approx 7 \times 10^8\,m$

$1.1 R_\odot \lesssim R \lesssim 4.8 R_\odot \Rightarrow e$ = eccentricity $\rightarrow$ large

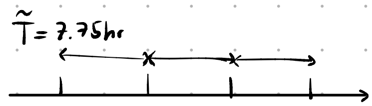
Kepler $\rightarrow$ ellipse



"aphelion" → "apastron"

"perihelion" → "periastron"

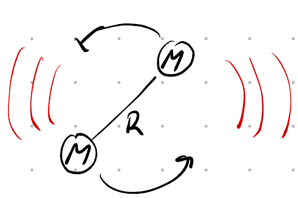
Observed over a decade:



$\tilde{T}$ decreases by $76\,\mu s = 7.6 \times 10^{-5}\,s$ per year

Therefore, we observe energy loss, which is NOT accounted for by Newtonian physics.

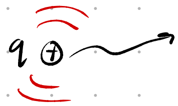
However, General Relativity (GR) predicts Gravitational Waves (GW):



Power $P = -\frac{2}{5}\frac{G^4 M^5}{R^5 c^5}$, G = Newton's constant $\approx 6.67 \times 10^{-11} \frac{m^3}{kg\,s^2}$

(Later on $G = 1$)

E & M : sychronic radiation



acceleration a , $P = -\frac{2}{3}\left(\frac{q^2}{4\pi\varepsilon_0}\right)\frac{a^2}{c^3}$ → similar to above

Solar system $P \approx -200\,W$

Hulse-Taylor $P = -7 \times 10^{24}\,W$

1983 : Hulse - Taylor announce indirect detection of GW
1993 : Nobel Prize !

2015 : LIGO direct detection $P = 3.6 \times 10^{49}\,W$
2017 : Nobel Prize : Barish, Weiss, Thorne

Newtonian Gravitation

$$\vec{F}_i = m_i \frac{d^2\vec{r}_i}{dt^2} \quad , \quad \vec{F}_i = \sum_{j \neq i} \frac{G m_i m_j}{|\vec{r}_i - \vec{r}_j|^3}(\vec{r}_j - \vec{r}_i)$$

These are instantaneous! We can say same thing about Coulomb's Law...

The solution in E&M is → Fields.

Aside: (Gaussian Units)

$\vec{E}, \vec{B}$ same units , $\vec{F} = q(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B})$

Maxwell's Equations

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$$

Try to fix gravity by combining with E&M :

Gravitoelectromagnetism Equations

$$\vec{\nabla}\cdot\vec{E}_G = -4\pi\rho_G \qquad \vec{\nabla}\times\vec{E}_G = -\frac{1}{c}\frac{\partial\vec{B}_G}{\partial t}$$

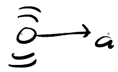
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B}_G = \frac{1}{c}\frac{\partial\vec{E}_G}{\partial t} - \frac{4\pi}{c}\vec{J}$$

Problem with the sign change is energy conservation: $u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2$ (SI)

$$\mathcal{E} = -\frac{1}{8\pi}\left(\vec{E}_G^{\,2} + \vec{B}_G^{\,2}\right)dV + \sum_i K_i$$

kinetic energy

negative energy leads to instability :
radiation creates more negative energy.



Lec 2 - 1/18 Special Relativity (SR)

- Notation / Terminology
- Hydrodynamics

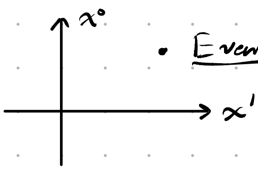
2:14 PM: STRING THEORY MENTIONED!!!
WHAT IS $\mathbb{R}^4$????

Terminology

spacetime (ct, x, y, z)

$$x^0 \equiv ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

spacetime diagram (dgm)



- Event = point in spacetime

coordinate system assigns 4 #'s to every event $\rightarrow (x^0, \underbrace{x^1, x^2, x^3}_{\text{space}})$

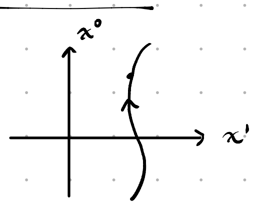
units of time $\rightarrow$ 1 m/c

Reference Frame (RF): same as coord. system

Observer: Same as RF

Inertial RF: Newton's 1^{st} Law holds
$F = 0 \rightarrow x = v_1 t, \; y = v_2 t, \; z = v_3 t$

World Line: Set of events that describe a particle



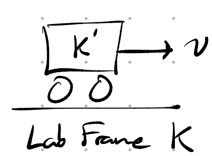
Events denoted by $A, B, C, \ldots$

spacetime coords x^α, $\alpha = 0, 1, 2, 3$
$\beta, \gamma, \delta, \ldots$

space-only coords: x_a $a, b, c, \ldots = 1, 2, 3$

In a particular RF: Can describe A

$K \swarrow$ x^α $\quad$ $K' \searrow$ $x^{\alpha'}$



RF denoted $K, K', K'', \ldots$

Postulates of SR

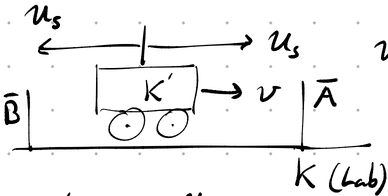
① Laws are same in all inertial RFs

② Speed of light $c = 3\times10^8\ \mathrm{m/s}$ in all inertial RFs

③ Isotropy - all directions are same

④ Homogeneity

1842 Doppler Effect



u_s = speed of sound (in K)

f' $\quad \lambda' = \frac{u_s}{f'}$ = on car
freq $\quad$ wavelength

At $\bar{A}$ (ahead of car)

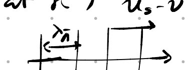
$$f_{\bar{A}} = f'\left(\frac{u_s}{u_s - v}\right), \quad \lambda_{\bar{A}} = \lambda'\left(\frac{u_s - v}{u_s}\right)$$

At $\bar{B}$ (behind car)

$$f_{\bar{B}} = f'\left(\frac{u_s}{u_s + v}\right), \quad \lambda_{\bar{B}} = \lambda'\left(\frac{u_s + v}{u_s}\right)$$

↳ Derivation: On the car (at K')

speed of sound (Ahead):



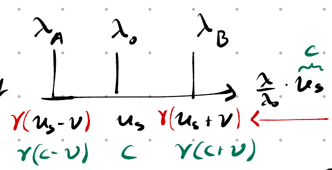
$$f'\lambda_{\bar{A}} = u_s - v \Rightarrow \lambda_{\bar{A}} = \frac{u_s - v}{f'} = \left(\frac{u_s - v}{u_s}\right)\lambda'$$

Not the case in SR, because "$u_s - v = u_s$" (no change of medium speed)

Relativistic Doppler effect

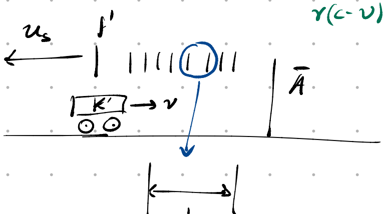
1938 Ives - Stillwell

Classical theory



← relativistic effect from *

should observe source wavelength ~~not~~ in between ahead and behind wavelengths. Test is Ives-Stillwell



$$v\Delta t = \frac{v}{f}$$

*Nuance: In relativity, there should be a scaling factor of $\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}$ due to time dillation

↳ $f = \frac{f'}{\gamma}$

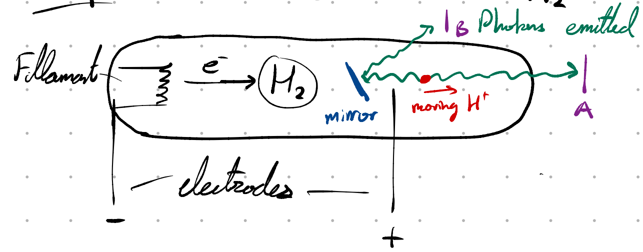
Ives - Stillwell : Moving H-atom

$n' = 4 \longrightarrow n = 2$ (Balmer Series) $\lambda = 4861\,\mathring{A}$ (blue)

Measured speed of H : $v = 0.005c$

Difference : $(\gamma-1)(c \pm v) \approx (\gamma-1)c$, $(\gamma-1)\, 4861\mathring{A} = 0.06\,\mathring{A}$

Setup: Vacuum Tube filled with H_2

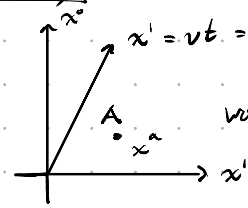


Electron Collision: $H_2 \rightarrow H_2^+ \longrightarrow H + H^+$

Compare frequency at A and B : ✓ detected difference

Back to theory

$\underline{K}$:



$x^1 = vt = \frac{v}{c} x^0$, $\beta = \frac{v}{c}$

worldline: $x^1 = \beta x^0$ A coords are x^α

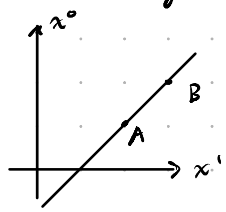
K' = RF of observer, $x^{\alpha'} = \begin{pmatrix} x^{0'} \\ x^{1'} \end{pmatrix}$ coords of A in K'

Homogeneity $\Rightarrow$ $x^{\alpha'}$ = linear expression in x^0, x^1

Light ray

For a light ray $x^1 = x^0$ and $x^{1'} = x^{0'}$.

($\underline{3D}$: Light ray through origin : $ct = \sqrt{x^2+y^2+z^2}$)



$$\Delta x^\alpha \equiv x^\alpha_B - x^\alpha_A$$
$$c\Delta t = ct_B - ct_A, \text{ \&c.}$$

By pythagorean thm, $\boxed{(c\Delta t)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2}$ for any two events along light ray.

Need to find

$$\underset{K}{x^\alpha} \longrightarrow \boxed{\underset{}{\overset{\text{"Lorentz"}}{\text{Transformation}}}} \longrightarrow \underset{K'}{x^{\alpha'}}$$

$\underline{\text{Linear}^* \Rightarrow \text{Matrix}}$

$$\begin{pmatrix} \Delta x^{0'} \\ \Delta x^{1'} \\ \Delta x^{2'} \\ \Delta x^{3'} \end{pmatrix} = \begin{pmatrix} \square\square\square\square \\ \\ 4\times4 \\ \text{Matrix} \end{pmatrix} \begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix}$$

$$\Delta x^{\alpha'} = \Lambda^{\alpha'}_0 \Delta x^0 + \cdots + \Lambda^{\alpha'}_3 \Delta x^3 = \sum_{\alpha=0}^{3} \Lambda^{\alpha'}_\alpha \Delta x^\alpha$$

$*$ $x^{0'} = f_0(x^0, x^1, x^2, x^3)$

$x^{1'} = f_1(\cdots \quad), \ldots$

$$\Rightarrow \Delta x^{0'} = \frac{\partial f_0}{\partial x^0} \Delta x^0 + \cdots + \frac{\partial f}{\partial x^3} \Delta x^3$$

$$\Rightarrow \begin{pmatrix} \Delta x^{0'} \\ \vdots \end{pmatrix} = \begin{pmatrix} 4\times4 \end{pmatrix} \begin{pmatrix} \Delta x^0 \\ \vdots \end{pmatrix},$$

where nonlinear if $\frac{\partial f}{\partial x^\alpha}$ aren't const.

If that were the case, then we could violate homogeneity (test the Lorentz transformation to determine location)

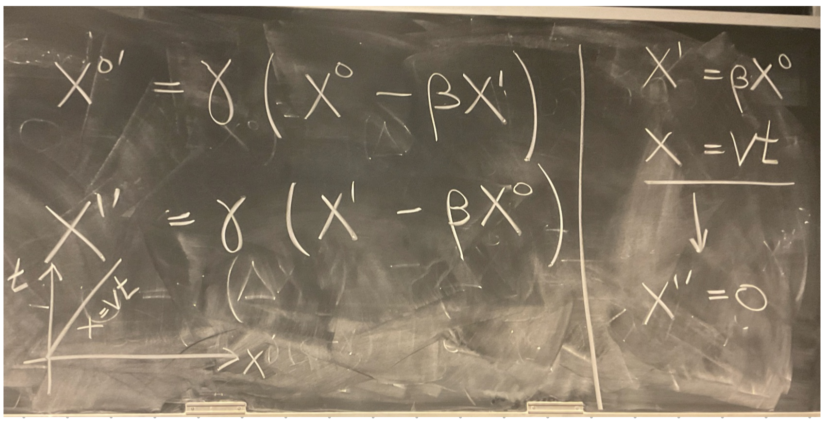
Cond of Light Ray

$$K: \; 0 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$$

$$K': \; 0 = (\Delta x^{0'})^2 - (\Delta x^{1'})^2 - (\Delta x^{2'})^2 - (\Delta x^3)^2$$

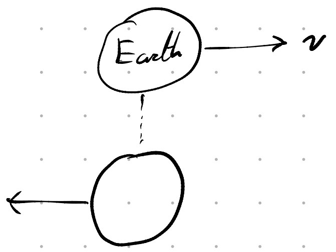
<u>Soln</u>: Lorentz transformations

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$



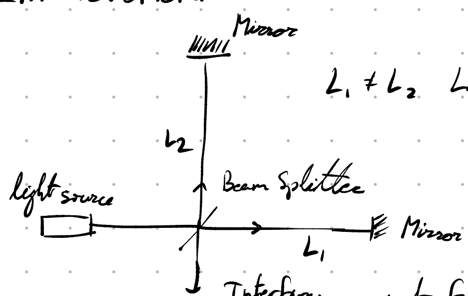
Lec 3 - 1/23

Michelson - Morley expt.



Might expect $c \to c \pm v$ as Earth moves through the aether

Interferometer

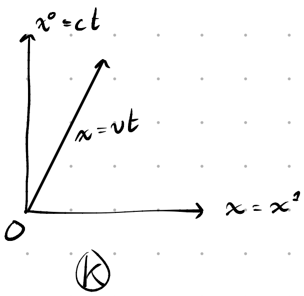


$L_1 \neq L_2$ Lorentz contraction

Interference not found.

Lorentz Transformations

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad \gamma = f(v)$$
$$x' = \gamma(x - vt)$$



(K) Worldline of light ray through origin O

$$\boxed{-x = ct} \xleftrightarrow{?} \boxed{-x' = ct'}$$

$$x' = ct' \Rightarrow -\cancel{\gamma}(x - vt) = \cancel{\gamma}c\left(t - \frac{v}{c^2}x\right)$$

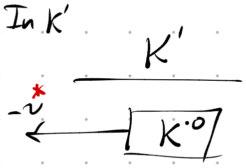
$$\Rightarrow \left(-1 + \frac{v}{c}\right)x = (c+v)t$$

$$\Rightarrow -\frac{c+v}{c}x = (c+v)t \Rightarrow -x = ct \quad ✓✓$$

Backwards is "showing" L.T.'s

Isotropy - All directions are the same

In K'



*Why $-v$? $x = 0$ (origin) $\to \begin{cases} t' = \gamma t \\ x' = -\gamma v t \Rightarrow x' = -vt' \end{cases}$

$$K \xrightarrow{?} K'$$

Method 1: Algebraic

$$\left.\begin{aligned} \gamma t - \gamma \frac{v}{c^2} x &= t' \\ -\gamma v t + \gamma x &= x' \end{aligned}\right\} \text{ solve for } x, t \Rightarrow \left.\begin{aligned} x &= \frac{x' + vt'}{\gamma\left(1 - \frac{v^2}{c^2}\right)} \\ t &= \frac{t' + \frac{v}{c^2}x'}{\gamma\left(1 - \frac{v^2}{c^2}\right)} \end{aligned}\right\}$$

Method 2: Isotropy

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \quad \text{Comes from interchanging } K \Leftrightarrow K' \text{ in L.T.'s}$$

$$x = \gamma(x' + vt')$$

by comparison with M.1. $\Rightarrow \gamma = \dfrac{1}{\gamma\left(1 - \frac{v^2}{c^2}\right)} \Rightarrow \boxed{\gamma = \dfrac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}}$ "γ-factor"

Notation

$\beta \equiv \frac{v}{c}$ dim'less

Lorentz transform's in spacetime: $\left.\begin{aligned} x^{0'} &= \gamma x^0 - \gamma\beta t \\ x^{1'} &= -\gamma\beta x^0 + \gamma x^1 \end{aligned}\right\}$ L.T.'s

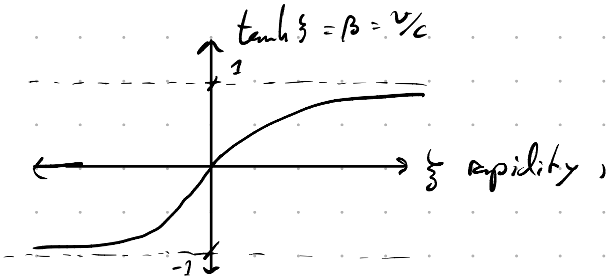
$\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $1 \leq \gamma$

observe: $\gamma^2 - (\gamma\beta)^2 = 1$

Reminds of $\cosh^2\xi - \sinh^2\xi = 1$

$$\cosh\xi \equiv \frac{e^{\xi} + e^{-\xi}}{2}, \quad \sinh\xi \equiv \frac{e^{\xi} - e^{-\xi}}{2}, \quad \tanh\xi = \frac{\sinh\xi}{\cosh\xi} = \frac{e^{\xi} - e^{-\xi}}{e^{\xi} + e^{\xi}}$$

Find ξ "Rapidity": $\cosh\xi = \gamma$, $\sinh\xi = \beta\gamma$ $\longrightarrow$ $\beta = \tanh\xi$



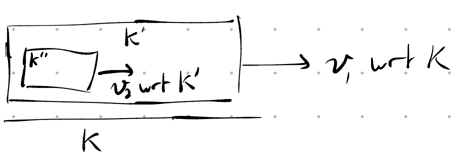
$-\infty < \xi < \infty$

$-1 < \frac{v}{c} < 1$

Leads to "Coordinate transform Matrix"

$$\underbrace{\begin{pmatrix} x^{0'} \\ x^{1'} \end{pmatrix}}_{X'} = \underbrace{\begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}}_{X}$$

Useful for: Addition of velocities $K \to K''$



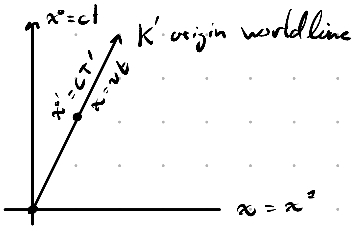
$$X'' = \Lambda_2 X' = \underbrace{\Lambda_2 \Lambda_1}_{\Lambda} X$$

$$\Rightarrow \begin{pmatrix} \cosh\xi_2 & -\sinh\xi_2 \\ -\sinh\xi_2 & \cosh\xi_2 \end{pmatrix} \begin{pmatrix} \cosh\xi_1 & -\sinh\xi_1 \\ -\sinh\xi_1 & \cosh\xi_1 \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix} \overset{!!!}{\Longrightarrow} \xi = \xi_1 + \xi_2$$

Id's
$$\cosh\xi_2 \cosh\xi_1 + \sinh\xi_2 \sinh\xi_1 = \cosh(\xi_1 + \xi_2)$$
$$\cosh\xi_2 \sinh\xi_1 + \sinh\xi_2 \cosh\xi_1 = \sinh(\xi_1 + \xi_2)$$

$$\frac{v_1}{c} = \tanh\xi_1, \quad \frac{v_2}{c} = \tanh\xi_2, \quad v = c\tanh(\xi_1 + \xi_2) = c\,\frac{\tanh\xi_1 + \tanh\xi_2}{1 + \tanh\xi_1 \tanh\xi_2} \Rightarrow \boxed{v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}}$$

Time dilation



$$K' : (x^{0'}, x^{1'}) = (cT', 0) \xrightarrow{LT} K : (\gamma cT', \gamma v T')$$

$\Rightarrow \gamma T'$ is time passed in K

Rossi-Hall μ (muon) expt.

Muons (2^{nd} generation Lepton)

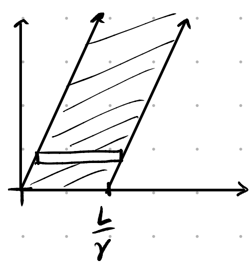
mass $\approx 106 \frac{MeV}{c^2}$ $\left(E = mc^2 \Rightarrow m = \frac{E}{c^2} \right)$

Lifetime $\longrightarrow \tau_\mu = 2.2 \times 10^{-6}$ s : Time T $\rightarrow e^{-T/\tau_\mu}$ remaining with $\boxed{SR}$: $\boxed{T}$ in K' (RF of muon)

In 2.2×10^{-6} s, $c\tau_\mu = 3 \times 10^8$ m/s $\times 2.2 \times 10^{-6}$ s $= 660$ m

$e^{-T/\gamma\tau_\mu}$ gives the expt.-ally correct half-life.

Lorentz Contraction



Ruler length L in K'

Events swept by ruler

In K' $\quad -\infty < t' < \infty$

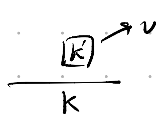
$\quad 0 \leq x' \leq L$

In K ? $\quad 0 \leq \gamma(x - vt) \leq L$, t fixed

$$\Rightarrow vt \leq x \leq \frac{L}{\gamma} + vt$$

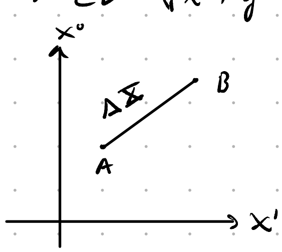
1D: Lightspeed const. $\Rightarrow x = ct \Leftrightarrow x' = ct'$

3D:



LS const $\rightarrow ct = \sqrt{x^2 + y^2 + z^2} \Rightarrow 0 = c t^2 - x^2 - y^2 - z^2$

Spacetime

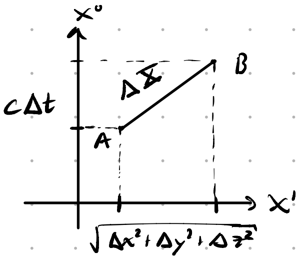


$$\Delta X = \begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^3 \\ \Delta x^4 \end{pmatrix} = \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Spacetime Interval: $\Delta s^2 = \underbrace{\Delta x^2 + \Delta y^2 + \Delta z^2}_{[\text{spatial distance}]^2} - c^2 \Delta t^2$

For A = origin, B = any event on worldline of lightray

then $\boxed{\Delta s^2 = 0}$



3 kinds of pairs of events

- Time-like separated : $\Delta s^2 < 0 \Rightarrow c\Delta t > \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$
- Space-like separated : $\Delta s^2 > 0 \Rightarrow c\Delta t < \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$
  - There is RF K′ for which $\Delta t' = 0$
- Null separated : $\Delta s^2 = 0 \Rightarrow$ Light ray can pass through A, B

Lorentz Transformations

$$\underset{(K')}{\Delta \underline{X}'} = \Lambda \underset{(K)}{\Delta \underline{X}}$$

$\Delta s'^2 = \Delta s^2 \rightarrow$ Actually, we only know $\Delta s^2 = 0 \leftrightarrow \Delta s'^2 = 0$

But, Isotropy:

$$\Delta s'^2 = \phi(v) \Delta s^2$$

$$\Delta s^2 = \phi(-v) \Delta s'^2 \Rightarrow \Delta s^2 = \phi(v)\phi(-v) \Delta s^2$$

$$\phi(v) = \phi(-v) \Rightarrow \phi^2 = 1 \Rightarrow \phi = \pm 1 \Rightarrow \phi = 1$$

↳ − soln implies time reversal.

$$\Delta s^2 = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 - (\Delta x^0)^2$$

$$= (\Delta x^0 \; \Delta x^1 \; \Delta x^2 \; \Delta x^3) \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix} = (\Delta x^0 \; \Delta x^1 \; \Delta x^2 \; \Delta x^3) \begin{pmatrix} -\Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix}$$

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ "Minkowski metric"

$$\Delta s^2 = \Delta X^T \eta \Delta X$$

Lorentz transforms : $\Delta X' = \Lambda \Delta X$

$$\Delta s^2 = \Delta s'^2 = \Delta X'^T \eta \Delta X' = \Delta X^T \Lambda^T \eta \Lambda \Delta X$$

$\Rightarrow$ For all ΔX : $\Delta X^T \Lambda^T \eta \Lambda \Delta X = \Delta X^T \eta \Delta X \Rightarrow \Lambda^T \eta \Lambda = \eta$

4 types of Λ's which solve:

LT's. $\begin{pmatrix} 0 & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{pmatrix}$

Parity, Time reversal

π Θ

Rots: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Lec 4 - 1/25

Lorentz Group

Spacetime Interval $(c\Delta t, \Delta x, \Delta y, \Delta z)$

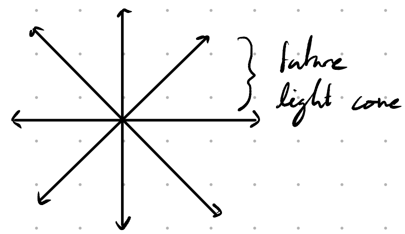
$$\Delta s^2 = -c\Delta t^2 + \underbrace{\Delta x^2 + \Delta y^2 + \Delta z^2}_{\text{distance}^2 \text{ between 2 events}}$$

$\Delta s^2 > 0$: space-like separated

$\Delta s^2 < 0$: time-like separated

$\Delta s^2 = 0$ null

If we take $A \to (0,0,0,0)$
$B \to (ct, x, y, z)$



$\Delta s^2 = 0$ Quadratic $\Rightarrow$ locus light cone (around A)

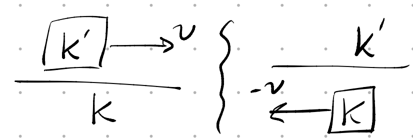
Lorentz transformation Λ

$$\underbrace{\begin{pmatrix} \Delta x^{0'} \\ \Delta x^{1'} \\ \Delta x^{2'} \\ \Delta x^{3'} \end{pmatrix}}_{\Delta \underline{X}'} = \underbrace{\begin{pmatrix} & \text{col } \alpha & \\ \text{row } \alpha' & \Lambda^{\alpha'}{}_{\alpha} & \\ & & \end{pmatrix}}_{\Lambda\ (4\times 4)} \underbrace{\begin{pmatrix} \Delta x^{0} \\ \Delta x^{1} \\ \Delta x^{2} \\ \Delta x^{3} \end{pmatrix}}_{\Delta \underline{X}}$$

$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$

$\Delta s'^2 = -c^2 \Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2$

$\Delta s^2 = \Delta s'^2$ is required of Λ Pf:



$\Delta s'^2 = \phi(v)\,\Delta s^2$, $\Delta s^2 = \phi(-v)\,\Delta s'^2$

Isotropy: $\phi(v) = \phi(-v) \to \Delta s^2 = \phi(v)^2 \Delta s^2$

$\Rightarrow \phi(v) = \pm 1 \to 1$

$$\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$ diagonal matrix "metric"

$\Delta \underline{X}' = \Lambda(\Delta \underline{X})$, $\Delta s^2 = \Delta \underline{X}^T \eta \Delta \underline{X}$

$\Delta s^2 = -c^2 t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$

$\Delta s'^2 = \Delta \underline{X}'^T \eta \Delta \underline{X}' = \Delta \underline{X}^T \Lambda^T \eta \Lambda \Delta \underline{X} \Rightarrow \boxed{\Lambda^T \eta \Lambda = \eta}$

Component notation

$$\Delta \underline{X}^{\alpha'} = \sum_{\alpha=0}^{3} \Lambda^{\alpha'}{}_{\alpha} \Delta \underline{X}^{\alpha} \longrightarrow \Lambda^{\alpha'}{}_{\alpha} \Delta \underline{X}^{\alpha}$$

Einstein Summation convention:

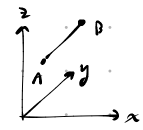
Rule: Include $\sum_{\gamma=0}^{3}$ for any index γ that repeats twice (IN PRODUCT)

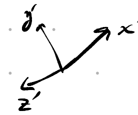
$$\eta_{\alpha\beta} = \begin{cases} -1 & \text{if } \alpha = \beta = 0 \\ +1 & \text{if } \alpha = \beta = 1,2,3 \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

$$\Delta s^2 = \Delta \underline{X}^T \eta \Delta \underline{X} = \Delta \underline{X}^{\alpha} \eta_{\alpha\beta} \Delta \underline{X}^{\beta}$$

Group

3d rotations (Analogy) $\underbrace{\begin{pmatrix}\Delta x' \\ \Delta y' \\ \Delta z'\end{pmatrix}}_{\Delta \underline{X}'} = \underbrace{\begin{pmatrix} & 3\times 3 & \\ & & \\ & & \end{pmatrix}}_{\Omega} \begin{pmatrix}\Delta x \\ \Delta y \\ \Delta z\end{pmatrix}$





rotated coordinate system

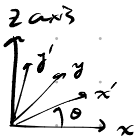
$\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$

$\Omega \rightarrow \Delta l'^2 = \Delta l^2$

Lin Alg: $\Delta l'^2 = \Delta \underline{X}'^T \Delta \underline{X}' = \Delta \underline{X}^T \Omega^T \Omega \, \Delta \underline{X} \overset{\text{should be}}{=} \Delta \underline{X}^T \Delta \underline{X}$

$\Rightarrow \boxed{\Omega^T \Omega = I}$

Soln



① Rotation

$\Omega = \begin{pmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ around $\hat{z}$ by angle θ, $\Omega = \begin{pmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{pmatrix}$ around $\hat{y}$

② Parity

$P = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$

$\Omega^T \Omega^T = I \rightarrow \Omega$ "orthogonal matrix" : Group $O(3)$: $\Omega_1, \Omega_2 \in O(3)$

$\Rightarrow \Omega = \Omega_1 \Omega_2 \in O(3)$
and $\Omega_2 \Omega_1 \in O(3)$

4D Solve $\Lambda^T \eta \Lambda = \eta$

$\rightarrow$ Lorentz Group $O(\underset{\text{time}}{1}, \underset{\text{space}}{3})$

solutions : combinations of 4 things

① Boost $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

In x direction:

$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

In y direction:

$\Lambda = \begin{pmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

and z...

Note: $\det \Lambda = 1$

② Rotations

$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta & -s_\theta & 0 \\ 0 & s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ around z (x, y also)

$\det \Lambda = 1$

Improper Lorentz transf.

③ $P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ ④ $T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\det P = \det T = -1$

If only Rotations, Boosts then Proper Lorentz Group

$\Lambda^T \eta \Lambda = \eta \Rightarrow \det \Lambda = \pm 1 \longrightarrow$ require $\det \Lambda = +1 \Rightarrow$ special $SO(1,3)$

$SO(1,3)$ includes $PT = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

Note: $\begin{pmatrix} \Lambda^{0'}{}_0 & \cdot & \cdot & \cdot \\ \Lambda^{1'}{}_0 & \cdot & \cdot & \cdot \\ \Lambda^{2'}{}_0 & \cdot & \cdot & \cdot \\ \Lambda^{3'}{}_0 & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \Lambda^{0'}{}_0 \\ \Lambda^{1'}{}_0 \\ \Lambda^{2'}{}_0 \\ \Lambda^{3'}{}_0 \end{pmatrix}$

$$-(\Lambda^{0'}{}_0)^2 + \underbrace{(\Lambda^{1'}{}_0)^2 + (\Lambda^{2'}{}_0)^2 + (\Lambda^{3'}{}_0)^2} = -1^2 + 0^2 + 0^2 + 0^2 = -1$$

$$\Rightarrow (\Lambda^{0'}{}_0)^2 = 1 + \text{positive or } 0 \geq 1$$

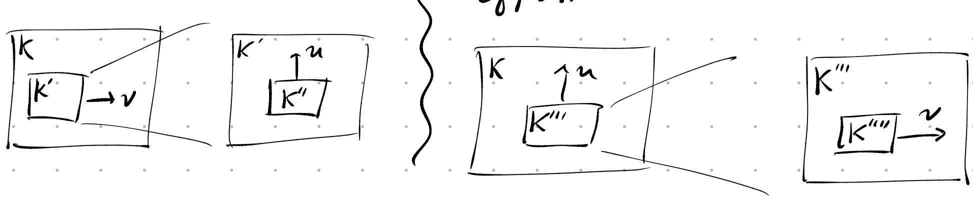
Require $\Lambda^{0'}{}_0 > 0$ & $\det \Lambda = 1 \Rightarrow SO^+(1,3)$

Infinitesimal Λ (Lie Algebra theory)

$\Lambda_1 = I + A_1$

$\Lambda_2 = I + A_2$

opposite order



$K'' \neq K'''$, but $\Lambda'' = \Lambda''' \cdot \Lambda_{rotation}$ "Thomas-Wigner Rotation"

$\Lambda_1 \Lambda_2 = (I + A_1)(I + A_2) = I + A_1 + A_2 + A_1 A_2$

$\Lambda_2 \Lambda_1 = I + A_1 + A_2 + A_2 A_1 \Rightarrow A_1 A_2 - A_2 A_1 =: [A_1, A_2]$ commutator of matrices

Rotations around $\hat{z}$

$$\Lambda = \begin{pmatrix} 1 & & & \\ & c_\theta & -s_\theta & \\ & s_\theta & c_\theta & \\ & & & 1 \end{pmatrix} = I + \begin{pmatrix} 0 & & & \\ & 0 & -\theta & \\ & \theta & 0 & \\ & & & 0 \end{pmatrix} \qquad {}^{*}c_\theta \to 1, \quad s_\theta \to \theta$$

$$S_3 := \begin{pmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix} \qquad \Lambda = I + \theta S_3$$

Boosts In x dir, small $\beta \Rightarrow \gamma \to 1$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = I + \begin{pmatrix} 0 & -\beta & & \\ -\beta & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Infinitesimal Boost

$$\left.\begin{aligned} \Lambda_1 &= I + \beta_1 K_1 \\ \Lambda_2 &= I + \beta_2 K_2 \end{aligned}\right\} \quad \begin{aligned} &\Lambda_1 \Lambda_2 \neq \Lambda_2 \Lambda_1 \\ &[K_1, K_2] = -S_3 \end{aligned} \quad \Rightarrow \text{Two boosts form a rotation}$$

Tensors

4-vectors

$$v^\alpha = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} \text{ in } K \longrightarrow v^{\alpha'} = \Lambda^{\alpha'}{}_{\alpha} v^\alpha \text{ in } K'$$

4-momentum

$$p^\alpha : \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} \left\{ \begin{aligned} &E'/c = \gamma\left(\frac{E}{c} - \beta p_x\right) \\ &\text{in } K' \\ &ct' = \gamma(ct - \beta x) \end{aligned} \right.$$

Next week ...

$p^\alpha = m\, u^\alpha$, see how above happens through coordinates

m: rest mass, u^α: 4-velocity

Lec 5 - 1/30   4-momentum, Tensors

K	K'
ϕ invariant	$\phi' = \phi$

Example

$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2 \equiv \Delta s^2 \quad \text{Spacetime interval}$$

$$= -\text{"(proper time)"}$$

4-Vector

K   K' $K \to K'$

$$\begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} \quad \begin{pmatrix} v^{0'} \\ v^{1'} \\ v^{2'} \\ v^{3'} \end{pmatrix}$$

$V^{\alpha'} = \Lambda^{\alpha'}_{\ \alpha} V^{\alpha}$ Λ Lorentz Transformation

$\alpha' = 0', 1', 2', 3'$ Transform Like Δx^{α}

Example 4-momentum

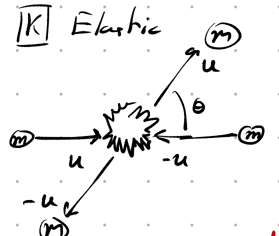
$$p^{\alpha} \to \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Notation Point Particle $\boxed{u}$

K: $m \bullet\!\!\longrightarrow u$ (rest mass), $\boxed{K'} \to v$ In K': $w = \dfrac{u - v}{1 - \frac{uv}{c^2}}$

If $u = 0$, $E = mc^2$. Why? Point: energy conversation

$\boxed{K}$ Elastic



$E \xrightarrow{?} \cancel{\tfrac{1}{2}mu^2}$ Bad!

$f(u) \Rightarrow \cdots$

K':

$$\frac{1}{2}m\left(\frac{u+v}{1+\frac{uv}{c^2}}\right)^2 + \frac{1}{2}m\left(\frac{u-v}{1-\frac{uv}{c^2}}\right)^2$$

$$\neq \frac{1}{2}m(\ )^2 + \frac{1}{2}m(\ )^2$$

angles velocity addition

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} = mc^2 + \frac{1}{2}mu^2 + mu^2\, O\!\left(\frac{u^2}{c^2}\right) + \cdots$$

$$p_x = \frac{m u_x}{\sqrt{1 - \frac{u^2}{c^2}}} = m \gamma_u u_x$$

Claim: With these expressions for E and p,

$$p^\alpha = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} \quad \text{is a 4-vector: } \quad p^{\alpha'} = \Lambda^{\alpha'}{}_\alpha \, p^\alpha$$

Check: $\boxed{K'} \longrightarrow v$

$m \bullet \longrightarrow u$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \Rightarrow \begin{pmatrix} p^{0'} \\ p^{1'} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \end{pmatrix}$$

Non-relativistic Limit (NR)

$|u| \ll c \quad |v| \ll c$ or $|\beta| \ll 1 \longrightarrow$

$\gamma \approx 1 \rightarrow \gamma = 1 + O(\beta^2)$

$$p^{1'} = \gamma(p^1 - \beta p^0) \qquad p^0 = \frac{E}{c} \approx mc + O(\beta)^2$$

$$\approx 1\left(mu - \left(\frac{v}{c}\right)mc\right) = m(u-v)$$

$$cp^{0'} = \gamma c(p^0 - \beta p^1) = \gamma(\underline{E} - \underline{v p^1})$$

$$\overset{?}{\approx} 1\left(mc^2 + \frac{1}{2}mu^2 - v\cdot mu\right) = mc^2 + \frac{1}{2}mu^2 - muv + \frac{1}{2}mv^2$$

BAD: $\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2}$

On the other hand: NR Mech

$$E' = cp^{0'} \longrightarrow \frac{1}{2}m(u-v)^2 + mc^2 = mc^2 + \frac{1}{2}mu^2 - muv + \frac{1}{2}mv^2 \; ?$$

1 2 3

Equations in Relativity

$$\underbrace{A^\alpha = B^\alpha}_{K} \text{ or } V^\alpha = 0 \longleftrightarrow \underbrace{V^{\alpha'} = 0}_{K'}$$

Comes from $\Lambda \cdot 0 = 0$

Angular Momentum

$L_z \rightarrow xp_y - yp_x$

$L_x \rightarrow yp_z - zp_x$

$$\Longrightarrow \boxed{L^{\alpha\beta} = x^\alpha p^\beta - x^\beta p^\alpha}$$

$(4\times 4 = 16 \text{ vals})$

$L^{\alpha\beta} = -L^{\beta\alpha}$ Anti-symmetric matrix (Tensor)

Angular Momentum Conservation

What else is conserved? $\qquad L^{\alpha\beta} = -L^{\beta\alpha}$

How many? 6 $\to$ angular mom. / $\to$?

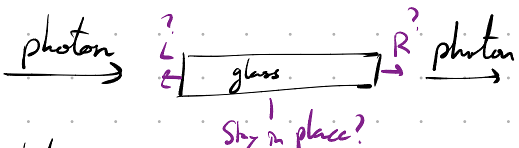
$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ - & & 0 & \\ & & & 0 \end{pmatrix}$$

$L^{12} = L_z,\ L^{23} = L_x$, etc.

$$L^{01} = x^0 p^1 - x^1 p^0 = ct \cdot p_x - \left(\frac{E}{c}\right) x \xrightarrow{NR} c\,[p_x t - m x]$$

Relativistically: $M_{TOT} \xrightarrow{R} \dfrac{E}{c^2} \quad \Longleftarrow \quad \sum_{\substack{\text{all} \\ \text{particles}}} \longrightarrow \vec{P}_{tot}\, t - M_{tot}\, \vec{x}_{cm} = \text{const}$ ✓

Useful to answer ?'s like:



photon $\longrightarrow$ early $\Rightarrow$ to compensate, glass moves left.

Two 4-vectors

$$A^\alpha,\ B^\alpha \longrightarrow \boxed{T^{\alpha\beta} = A^\alpha B^\beta}$$

Transformations: $K \to K'$

$$T^{\alpha'\beta'} = A^{\alpha'} B^{\beta'} = (\Lambda^{\alpha'}_{\ \alpha} A^\alpha)(\Lambda^{\beta'}_{\ \beta} B^\beta) = \Lambda^{\alpha'}_{\ \alpha} \Lambda^{\beta'}_{\ \beta} A^\alpha B^\beta = \Lambda^{\alpha'}_{\ \alpha} \Lambda^{\beta'}_{\ \beta} T^{\alpha\beta}$$

Rank-2 (Contravariant) tensor

$4 \times 4 = 16$ $T^{\alpha\beta}$ that transform like $\boxed{T^{\alpha'\beta'} = \Lambda^{\alpha'}_{\ \alpha} \Lambda^{\beta'}_{\ \beta} T^{\alpha\beta}}$

Example E&M $\quad [\vec{E}] = [v\vec{B}] \,/\, [E/c] = [B]$ ← units

$\vec{E}, \vec{B} \longrightarrow F^{\mu\nu}$

"Field Strength tensor"

$$F = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

$\boxed{K}$ B_z, E_y

$E_y, B_z \neq 0$

$\boxed{K'} \longrightarrow v$

$\underline{K'}$ $\quad E_y' = -c F^{0'2'}$

$$-\frac{E_y'}{c} = F^{0'2'} = \Lambda^{0'}_{\ \underset{0,1}{\alpha}} \Lambda^{2'}_{\ \underset{2}{\beta}} F^{\alpha\beta}$$

$$\Lambda^{0'}_{\ 0} = \gamma \qquad \Lambda^{0'}_{\ 1} = -\gamma\beta \qquad \Lambda^{2'}_{\ 2} = \Lambda^{3'}_{\ 3} = 1$$

$$\Lambda^{2'}_{\ 0} = -\gamma\beta \qquad \Lambda^{1'}_{\ 1} = \gamma \qquad \longrightarrow$$

$$\longrightarrow \quad -\frac{E_y'}{c} = F^{0'2'} = \Lambda^{0'}_{\ \alpha} \Lambda^{2'}_{\ \beta} F^{\alpha\beta} = \Lambda^{0'}_{\ \alpha} \Lambda^{2'}_{\ 2} F^{\alpha 2} = \Lambda^{0'}_{\ 0} \Lambda^{2'}_{\ 2} F^{02} + \Lambda^{0'}_{\ 1} \Lambda^{2'}_{\ 2} F^{12}$$

$$= \gamma F^{02} - \gamma\beta F^{12} = \gamma\left(-\frac{E_y}{c}\right) - \gamma\beta(-B_z) \Rightarrow E_y' = \gamma E_y - \gamma v B_z$$

Rank-n (Contravariant) tensor

4^n quantities $\quad T^{\alpha_1\alpha_2\cdots\alpha_n}$ $\quad \left(\sum_{\alpha_1\cdots\alpha_n}\right)$

Transformation Rule: $\boxed{T^{\alpha_1'\alpha_2'\cdots\alpha_n'} = \Lambda^{\alpha_1'}{}_{\alpha_1}\Lambda^{\alpha_2'}{}_{\alpha_2}\cdots\Lambda^{\alpha_n'}{}_{\alpha_n}T^{\alpha_1\alpha_2\cdots\alpha_n}}$

Covariant:

$K' \longrightarrow K \qquad \Lambda^{\alpha'}{}_{\alpha} \to \begin{pmatrix}\gamma & -\gamma\beta\\ -\gamma\beta & \gamma\end{pmatrix}$

$\Lambda^{\alpha}{}_{\alpha'}$ (inverse matrix) $\Lambda^{\alpha}{}_{\alpha'} \to \begin{pmatrix} & \\ & \end{pmatrix}^{-1} = \begin{pmatrix}\gamma & \gamma\beta\\ \gamma\beta & \gamma\end{pmatrix}$

Covariant 4-vector

$$\boxed{V_{\alpha'} = \Lambda^{\alpha}{}_{\alpha'}V_\alpha}$$

arises from: Gradient

$\phi' = ? \quad \begin{pmatrix}x^{0'}\\ \vdots\end{pmatrix}$

$\phi(x^0,\ldots,x^3) \leftarrow \boxed{\text{scalar}}$ field

$K': \phi'(x^{0'},\ldots,x^{3'}) = \phi(x^0(x^{0'},\ldots,x^{3'}),\ldots)$

$$\begin{pmatrix}\frac{\partial\phi}{\partial x^0}\\ \frac{\partial\phi}{\partial x^1}\\ \frac{\partial\phi}{\partial x^2}\\ \frac{\partial\phi}{\partial x^3}\end{pmatrix} \quad \frac{\partial\phi}{\partial x^\alpha} = \partial_\alpha\phi \text{ gradient}$$

$$K': \partial_{\alpha'}\phi = \frac{\partial\phi}{\partial x^{\alpha'}} = \frac{\partial x^\alpha}{\partial x^{\alpha'}}\left(\frac{\partial\phi}{\partial x^\alpha}\right) = \Lambda^{\alpha}{}_{\alpha'}\partial_\alpha\phi$$

$$x^\alpha = \Lambda^{\alpha}{}_{\alpha'}x^{\alpha'}$$

Rank-n (Covariant) tensor

$T^{\alpha_1\alpha_2\cdots\alpha_n}$ $\quad \left(\sum_{\alpha_1\cdots\alpha_n}\right)$

Transformation Rule: $\boxed{T_{\alpha_1'\alpha_2'\cdots\alpha_n'} = \Lambda^{\alpha_1}{}_{\alpha_1'}\Lambda^{\alpha_2}{}_{\alpha_2'}\cdots\Lambda^{\alpha_n}{}_{\alpha_n'}T_{\alpha_1\alpha_2\cdots\alpha_n}}$

Lec 6 - 2/1

Worldlines: 4-velocity, 4-acceleration, 4-momentum

- Covariant Tensors
- New tensors from old (Lowering/Raising)

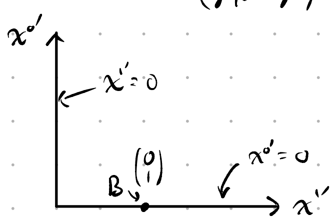
$\boxed{K'} \longrightarrow v$

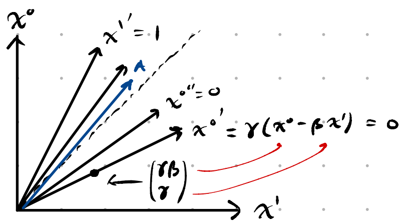
K

$K \longrightarrow K'$ $\qquad x^0 = \gamma(x^{0'} + \beta x^{1'})$

$\qquad\qquad\quad x^1 = \gamma(x^{1'} + \beta x^{0'})$

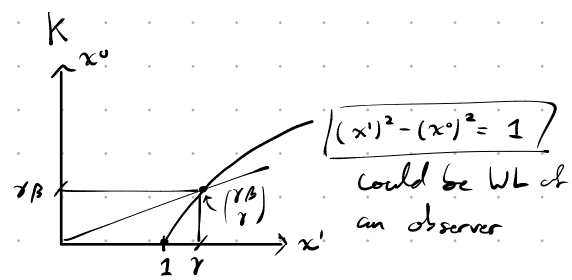
$$\Lambda^{\alpha}{}_{\alpha'} \longrightarrow \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}, \quad \Lambda^{0}{}_{1'} = \gamma\beta$$





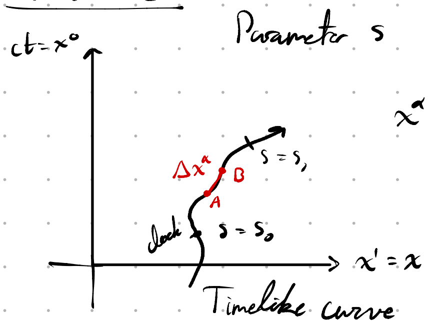
$\begin{pmatrix}\gamma\beta \\ \gamma\end{pmatrix}$ as $v = \beta c$ varies

$\Rightarrow \gamma^2 - (\gamma\beta)^2 = 1$



Worldlines

Parameter s



$x^\alpha(s)$: parameterize the WL

E.g.: $s = t$ = time (K)

$x^0 = cs$

$x^1 = vs$ (const velocity)

$x^2 = x^3 = 0$

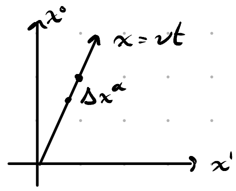
Good Parameter: τ = proper time

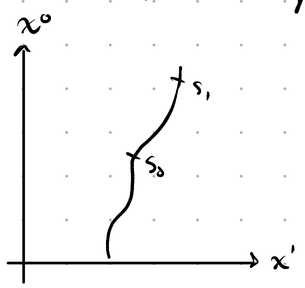
$$\Delta x^\alpha = x^\alpha_B - x^\alpha_A, \quad c\Delta\tau = \sqrt{-\Delta s^2} = \sqrt{(\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2}$$

Why? In K': $\Delta x^{1'} = \Delta x^{2'} = \Delta x^{3'} = 0 \rightarrow \Delta\tau = \sqrt{c^2 \Delta t'^2} = c\Delta t'$

MCRF$_{(s)}$ = K' (Momentarily Comoving Ref Frame)

$$\Delta\tau = \frac{1}{c}\sqrt{c^2\Delta t^2 - \Delta x^2}$$
$$= \frac{c\Delta t}{c}\sqrt{1-\frac{\Delta x^2}{c^2\Delta t^2}}$$
$$= \Delta t\sqrt{1-\beta^2} = \frac{\Delta t}{\gamma}$$





General
Any Timelike curve
WL

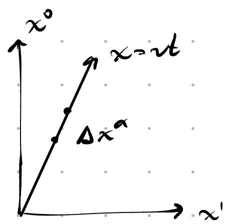
$$\Delta\tau = \int_{s_0}^{s_1}\frac{dt}{\gamma} = \int_{s_0}^{s_1}\frac{1}{\gamma(s)}\left(\frac{dt}{ds}\right)ds \quad , \quad \gamma = 1\Big/\sqrt{1-v(s)^2/c^2}$$

$\Rightarrow$ <u>change parameterization</u>

$$\tau(s) = \int_{s_0}^{s}\frac{dt}{\gamma} \Rightarrow \text{Inverse: } s = f(\tau)$$
$$x^\alpha(s) = x^\alpha(f(\tau)) = x^\alpha(\tau)$$

$$\Rightarrow \frac{\Delta x^\alpha}{\Delta\tau} \longrightarrow \boxed{u^\alpha \equiv \frac{dx^\alpha}{d\tau} \quad \text{4-vector "4-velocity"}}$$

<u>First</u> $(\beta = \frac{v}{c})$, $\gamma = 1/\sqrt{1-\beta^2}$



$$\begin{cases} x^0 = cs & t = s \\ x' = vs & \gamma = \text{const} \end{cases}$$
$$\tau = \int_0^s \frac{dt}{\gamma} = \frac{s}{\gamma}$$

$$\Rightarrow x^0 = c\gamma\tau, \quad x' = v\gamma\tau$$

$$\Rightarrow \text{4-velocity: } u^\alpha \longrightarrow \begin{pmatrix}\gamma c \\ \gamma v\end{pmatrix}$$

$$\tau = \gamma\left(t - \frac{v}{c^2}x\right)$$
$$\Rightarrow \frac{d\tau}{dt} = \gamma\left(1-\frac{v^2}{c^2}\right) = \frac{\gamma}{\gamma^2} = \frac{1}{\gamma}$$
$$\Rightarrow \frac{dt}{d\tau} = \gamma$$

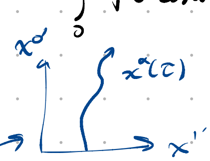
<u>Ex 2</u>

$$x^0 = b\sinh\sigma \quad , \quad \cosh^2\sigma - \sinh^2\sigma = 1$$
$$x' = b\cosh\sigma$$

$$c\tau = \int_0^\sigma\sqrt{-ds^2} = \int_0^\sigma\sqrt{(dx^0)^2-(dx')^2} = \int_0^\sigma\sqrt{\left(\frac{dx^0}{d\sigma}\right)^2-\left(\frac{dx'}{d\sigma}\right)^2}\,d\sigma$$

Spacetime Interval
Not same as σ-parameter

$$= \int_0^\sigma\sqrt{b^2\cosh^2\sigma - b^2\sinh^2\sigma}\,d\sigma = b\sigma$$



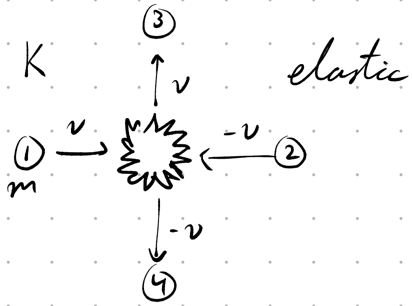
$$\Rightarrow c\tau = b\sigma \Rightarrow \sigma = \frac{c\tau}{b}$$

$$x^\alpha(\tau): \begin{cases} x^0 = b\sinh\left(\frac{c\tau}{b}\right) \\ x' = b\cosh\left(\frac{c\tau}{b}\right)\end{cases} \Rightarrow \text{4-velocity: } u^\alpha = \frac{dx^\alpha}{d\tau}$$

m = rest mass of particle $\Bigg\} \Rightarrow \boxed{p^\alpha \equiv m u^\alpha \text{ "4-momentum"}}$

$$p^\alpha = \begin{pmatrix} \gamma m c \\ \gamma m \overset{\frac{dx}{dt}}{v} \end{pmatrix} \xrightarrow[|\beta| \ll 1]{NR} \approx \begin{pmatrix} \frac{mc^2 + \frac{1}{2}mv^2}{c} \\ mv \end{pmatrix} = \begin{pmatrix} E/c \\ p_x \\ \vdots \end{pmatrix}$$

<u>Conservation</u>



$$p_1^\alpha + p_2^\alpha = p_3^\alpha + p_4^\alpha$$

b/c p_i^α is 4-vector,
it <u>is definitely</u> invariant,
i.e. conserved ✓

<u>Recall</u>: 4-vectors transform like:

$$v^{\alpha'} = \Lambda^{\alpha'}{}_{\alpha}\, v^\alpha$$

<u>Also</u>: $u^\alpha(\tau) = \dfrac{dx^\alpha}{d\tau} \Rightarrow$

$$\boxed{\begin{array}{ll} a^\alpha := \dfrac{du^\alpha}{d\tau} & \text{4-acceleration} \\ f^\alpha := m a^\alpha & \text{4-force} \end{array}}$$

<u>Tensors</u>

contrav. 4-vectors

$$\boxed{V^{\alpha'} = \Lambda^{\alpha'}{}_{\alpha} V^\alpha}$$

cov. 4-vectors

$$\boxed{W_{\alpha'} = \Lambda^{\alpha}{}_{\alpha'} W_\alpha} \quad \text{e.g. } \frac{\partial \phi}{\partial x^\alpha} = \partial_\alpha \phi$$

$$\left\{ \begin{array}{l} \Lambda^{\alpha'}{}_{\alpha} \rightarrow \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \\ \text{Inverse}: \Lambda^{\alpha}{}_{\alpha'} \rightarrow \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \\ \qquad \text{matrix} \rightarrow \Lambda^{-1} \end{array} \right.$$

$$\left. \begin{array}{l} A^\alpha \text{ contrav.} \\ B_\alpha \text{ cov.} \end{array} \right\} A^\alpha B_\alpha = \phi \quad \underline{\text{Invariant!}}$$

$$\text{In } K'?: \phi' = A^{\alpha'} B_{\alpha'} = (\Lambda^{\alpha'}{}_{\gamma} A^\gamma)(\Lambda^{\beta}{}_{\alpha'} B_\beta)$$

$$= \underbrace{\Lambda^{\beta}{}_{\alpha'} \Lambda^{\alpha'}{}_{\gamma}}_{\Lambda^{-1}\Lambda = I} A^\gamma B_\beta$$

$$\text{or} \quad \Lambda^{\beta}{}_{\alpha'} \Lambda^{\alpha'}{}_{\gamma} = \delta^\beta{}_\gamma = \begin{cases} 1 & \beta = \gamma \\ 0 & \beta \neq \gamma \end{cases} \quad \text{"Kronecker delta"}$$

$$\Rightarrow A^{\alpha'} B_{\alpha'} = \delta^\beta{}_\gamma B_\beta A^\gamma \underset{\delta^\beta{}_\gamma:\ \gamma = \beta}{=} A^\beta B_\beta = \phi \quad ✓$$

<u>Picture</u>: $\overset{\text{4-vector}}{A^\alpha} \longrightarrow \boxed{B_\alpha} \longrightarrow \phi = A^\alpha B_\alpha$

Covariant 2-Tensors

$$T_{\alpha'\beta'} = \Lambda^{\alpha}{}_{\alpha'} \Lambda^{\beta}{}_{\beta'} T_{\alpha\beta}$$

Mixed rank $\binom{1}{1}$ tensors

$$S^{\alpha'}{}_{\beta'} = \Lambda^{\alpha'}{}_{\alpha} \Lambda^{\beta'}{}_{\beta} S^{\alpha}{}_{\beta}$$

Contract indices

$S^{\alpha}{}_{\beta}$ $\qquad \phi = S^{\gamma}{}_{\gamma}$ $\qquad$ Check: $S^{\gamma'}{}_{\gamma'} = \Lambda^{\gamma'}{}_{\alpha} \Lambda^{\beta}{}_{\gamma'} S^{\alpha}{}_{\beta} = \delta^{\beta}{}_{\alpha} S^{\alpha}{}_{\beta} = S^{\alpha}{}_{\alpha}$

$\binom{1}{1} \longrightarrow$ scalar

$\binom{M}{N}$ tensor :

4^{M+N} components : $T^{\alpha_1\alpha_2\cdots\alpha_M}{}_{\beta_1\beta_2\cdots\beta_N}$

Transforms as : $\boxed{T^{\alpha_1'\alpha_2'\cdots\alpha_M'}{}_{\beta_1'\beta_2'\cdots\beta_N'} = \Lambda^{\alpha_1'}{}_{\alpha_1} \cdots \Lambda^{\alpha_M'}{}_{\alpha_M} \Lambda^{\beta_1}{}_{\beta_1'} \cdots \Lambda^{\beta_N}{}_{\beta_N'} T^{\alpha_1\alpha_2\cdots\alpha_M}{}_{\beta_1\beta_2\cdots\beta_N}}$

$$\binom{M+1}{N+1} \xrightarrow{\text{contraction}} \binom{M}{N} \text{ tensor}$$

$$T^{\alpha_1\cdots\alpha_M\gamma}{}_{\beta_1\cdots\beta_N\gamma} \underset{\sum_{\gamma=0}^{3}}{=} W^{\alpha_1\cdots\alpha_M}{}_{\beta_1\cdots\beta_N}$$

Tensor Product

$$\underset{\binom{M}{N}}{A^{\alpha_1\alpha_2\cdots\alpha_M}{}_{\beta_1\cdots\beta_N}} \; \underset{\binom{P}{Q}}{B^{\alpha_{M+1}\cdots\alpha_{M+P}}{}_{\beta_{N+1}\cdots\beta_{N+Q}}} \longrightarrow \overset{\binom{M+P}{N+Q}}{4^{M+N+P+Q} \text{ components}}$$

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \eta^{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \delta^{\alpha}{}_{\beta} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

2-tensor $\qquad$ 2-tensor $\qquad$ $\binom{1}{1} \to$ 2-tensor

$$P^{\alpha} = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} -E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\eta_{\alpha\beta} P^{\gamma} \xrightarrow{\text{contract}} \eta_{\alpha\gamma} P^{\gamma} = P_{\alpha}$$

Lec 7 - 2/6

- Tensors
- Relativistic Fluids

- MCRF (momentarily comoving RF)
- Energy-Momentum TENSOR
  stress-energy

Contravariant

$$T^{\alpha'\beta'} = \Lambda^{\alpha'}_{\ \alpha}\Lambda^{\beta'}_{\ \beta}T^{\alpha\beta}$$

$$\underset{K'}{\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix}} \xleftarrow{\Lambda} \underset{K}{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}$$

Covariant

$$W_{\alpha'\beta'} = \Lambda^{\alpha}_{\ \alpha'}\Lambda^{\beta}_{\ \beta'}W_{\alpha\beta}$$

$$K' \xleftarrow{\Lambda^{-1}} K$$

Mixed

$$S^{\alpha'}_{\ \beta'} = \Lambda^{\alpha'}_{\ \alpha}\Lambda^{\beta}_{\ \beta'}S^{\alpha}_{\ \beta}$$

Lowering/Raising α, β, γ covar. $\longleftrightarrow$ contrav.

Contraction

$$T^{\alpha}_{\ \beta} \rightarrow T^{\gamma}_{\ \gamma} = \sum_{\gamma=0}^{3} T^{\gamma}_{\ \gamma} = \text{trace of } T^{\alpha}_{\ \beta} = T$$

rank 2 (mixed)

3 Special Rank-2 tensors

$\eta^{\alpha\beta}$, $\delta^{\alpha}_{\ \beta}$, $\eta_{\alpha\beta}$
(inverse)

$$\eta_{\alpha\beta} \rightarrow \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \Delta s^2 = \eta_{\alpha\beta}\Delta x^{\alpha}\Delta x^{\beta}$$

Numbers are Invariant under RF

$$\boxed{c = 1 \quad \text{for the rest of the course}}$$

space & time measured in meters [m]

Algebraic conditions on Λ

$$\Lambda^T \eta \Lambda = \eta, \quad \eta_{\alpha'\beta'} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Claim: $\eta_{\alpha\beta}$ is a tensor: $\eta_{\alpha'\beta'} \stackrel{?}{=} \Lambda^{\alpha}_{\ \alpha'}\Lambda^{\beta}_{\ \beta'}\eta_{\alpha\beta} \xrightarrow{\text{matrices}} \eta = (\Lambda^{-1})^T\eta(\Lambda^{-1})$ ✓

$$\delta^\alpha_\beta = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$ also ✓

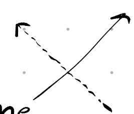
Lowering / Raising

V^α = 4-vector

$V_\alpha = \eta_{\alpha\beta} V^\beta$, $\eta_{\alpha\beta}$ is the metric

A^α contravariant, B_α covariant $\Rightarrow A^\alpha B_\alpha = \phi$ scalar invariant.

Example



me — particle

u^α — p^α, A^α

my 4-velocity

What's the energy $\mathcal{E}$ of particle in my MCRF?

Note: This question is Lorentz invariant, i.e., doesn't depend on RF, since it is a measurement made whose outcome must agree for everyone

$$\boxed{\mathcal{E} = -u^\alpha p_\alpha}$$ Ignore y, z

Derivation

$$u^\alpha = (\underset{\text{my } \gamma}{\gamma}, \gamma v) \xrightarrow[\text{MCRF}]{\text{my}} (1, 0)$$

$$p_\alpha = \begin{pmatrix} -E \\ p \end{pmatrix} \xrightarrow{\text{my MCRF}} \mathcal{E} \Rightarrow u^\alpha p_\alpha = -E \cdot 1 \rightarrow -\mathcal{E}$$

or $\mathcal{E} = -u^\alpha \eta_{\alpha\beta} p^\beta = -u \cdot p$ (dot product of 4-vectors)

How to measure $\vec{p}$ in MCRF

$$\begin{pmatrix} E \\ p \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ p \end{pmatrix}$$

$$p^\alpha - \mathcal{E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = p^\alpha - \mathcal{E} u^\alpha$$

A^0 in MCRF $\rightarrow -A_\alpha u^\alpha$

Example 2

Particle = Me, my Energy in MCRF?

$$m = -u^\alpha p^\alpha \Rightarrow m^2 = -p^\alpha p_\alpha$$

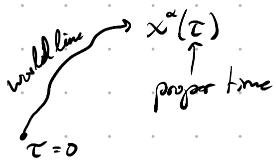
Math Aside:

$$A^\alpha B_\alpha = A_\alpha B^\alpha,$$

$$A^\alpha A_\alpha = -(A^0)^2 + \cdots + (A^3)^2$$

$$= \tilde{A} \cdot \tilde{A} = \tilde{A}^2$$

Point Particle



$$u^a = \frac{dx^a}{d\tau}$$

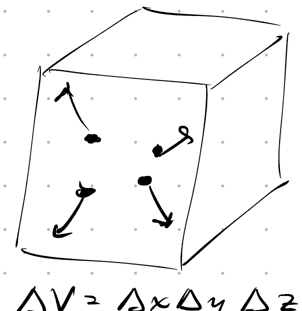
$$a^a = \frac{du^a}{d\tau}$$

Identity

$$u^a u_a = -1 \quad \text{Pf: In MCRF, } u^a \to \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\frac{d}{d\tau} \to 2u^\alpha \frac{du_\alpha}{d\tau} = 0 \Rightarrow u^a a_\alpha = 0 \Rightarrow a^\alpha \to \begin{pmatrix}0\\a^1\\a^2\\a^3\end{pmatrix}$$

Hydrodynamics : FLUIDS



$f(t, x, y, z)$ assume one-type atoms.

$\vec{v}(t, x, y, z)$ Non-relativistic (N.R.) velocity (3d)

n : particle density

$\frac{\Delta N}{\Delta V} \longrightarrow n$, ΔN = # particles in ΔV "the box"

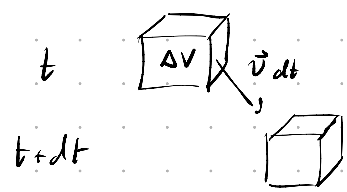
ρ : mass density, $\rho \leftrightarrow \frac{\Delta m}{\Delta V}$,

$\rho = mn$ N.R.

P = pressure $\longleftarrow \frac{\Delta F}{\Delta A}$ $\boxed{\Delta A}$

Problem

$a = \frac{dv}{dt} \Longrightarrow$ Fluid?



Convective derivative

$$\underset{\text{acceleration}}{\vec{a}} = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t, r + \vec{v}\Delta t) - \vec{v}(t, \vec{r})}{\Delta t} = \frac{\partial \vec{V}}{\partial t} + v^i \left(\frac{\partial \vec{v}}{\partial x^i}\right)$$

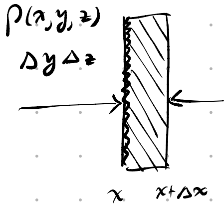
$$\boxed{\frac{Df}{Dt} \overset{\text{def}}{\equiv} \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f} \Rightarrow \vec{a} = \frac{D\vec{v}}{Dt}$$

Newton's Second Law

$$\rho \frac{D\vec{v}}{Dt} = \vec{f} = -\vec{\nabla} p \quad (-\text{gradient of pressure})$$

$$\Downarrow$$

$$\rho\, \partial_t v^i + \underbrace{v^j \partial_j}_{\vec{v}\cdot\vec{\nabla}} v^i = -\partial_i p$$



$p(x+\Delta x, y z)\,\Delta y \Delta z$

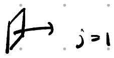
Net: $[p(x+\Delta x) - p(x)]\,\Delta y\,\Delta z \to -\left(\frac{\partial p}{\partial x}\right) \overbrace{\Delta x\,\Delta y\,\Delta z}^{\Delta V} = \overbrace{\rho\,\Delta V}^{\text{mass}}\, a_x$

$\Rightarrow\; -\nabla \vec{p} = \rho\,\vec{a}$

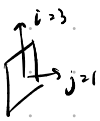
Stress tensor (3×3)

$\boxed{\Delta A}$ $\quad T_{ij} = \lim_{\Delta A \to 0} \frac{\text{Force}_i}{\Delta A \perp j}$

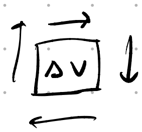
T_{11}



T_{31}



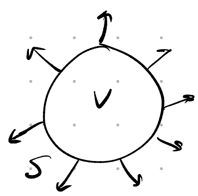
Argument Torque $\vec{r} \times \vec{F}$
$\propto \Delta V,\ L^3$



Moment of Inertia $MR^2 \propto L^5 \;\Rightarrow$ angular acc $\sim \frac{L^3}{L^5} \xrightarrow{L \to 0} \infty \;\Rightarrow$ no torque

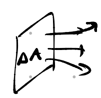
Conservation Laws

$n = \Delta N / \Delta V$



$\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot \vec{J}$

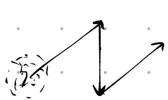
$$\frac{d}{dt} \iiint n\, dV = -\iiint (\vec{\nabla} \cdot \vec{J})\, dV = -\oint\!\!\oint_S \vec{J} \cdot d\vec{a}, \qquad \vec{J} = \frac{\Delta N}{\Delta A\, \Delta t} = n\vec{v}$$



Conservation of number of Particles

$$\boxed{\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot (n\vec{v})}$$

mean free path:



Lec 8 - 2/8

Lowering/Raising

$$V^\alpha \to \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \end{pmatrix}, \quad V_\alpha \longrightarrow \begin{pmatrix} -V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

$$\boxed{V^\alpha = \eta^{\alpha\beta} V_\beta} \qquad \boxed{V_\alpha = \eta_{\alpha\beta} V^\beta}$$

Raising index $\qquad$ Lowering index

$$T^{\alpha\beta} \to \begin{cases} T_\alpha{}^\beta = \eta_{\alpha\gamma} T^{\gamma\beta} \\ T^\alpha{}_\beta = \eta_{\gamma\beta} T^{\alpha\gamma} \\ T_{\alpha\beta} = \eta_{\alpha\gamma}\eta_{\beta\delta} T^{\gamma\delta} \end{cases}$$

$$\Rightarrow T_{00} = T^{00}$$
$$T_{01} = -T^{01}$$
$$T_{10} = -T^{10}$$
$$T_{11} = T^{11}$$

$$T^{\alpha\beta} \longrightarrow T_{\alpha\beta} \Rightarrow T^{\alpha\beta} T_{\alpha\beta} = (T^{00})^2 - \sum_{i=1}^{3} (T^{0i})^2 - \sum_{i=1}^{3} (T^{i0})^2 + \sum_{i,j}^{3} (T^{ij})^2$$

Hydrodynamics

Recap

NR = Nonrelativistic

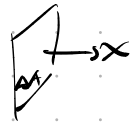
n = number density $\longleftarrow \dfrac{\Delta N}{\Delta V}$

ρ = mass density $= \dfrac{\Delta m}{\Delta V}$

$v^i \; (t, x, y, z)$

Flux/current of particles

$\vec{j}: \quad j_x = \dfrac{\Delta N}{\Delta A \cdot \Delta t}, \quad \Delta A = \Delta y \Delta z$



Convective Derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} = \partial_0 + v_i \partial_i$$

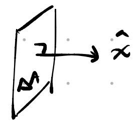
Conservation of number of Particles

$$0 = \frac{\partial n}{\partial t} + \vec{\nabla}\cdot\vec{J} = \partial_0 n + \partial_i j_i$$

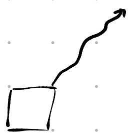
Fluid Dynamics

Stress (3D) Tensor

$$T^{ij} = \frac{\Delta F_i}{\Delta A \perp_j} \qquad T^{i1}:$$



Relativistic Fluid



ΔV

$x^\alpha(\tau)$: worldline of fluid element $\qquad \frac{dx^\alpha}{d\tau} = u^\alpha$ 4-velocity

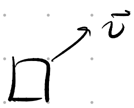
$u^\alpha(t,x,y,z)$ 4-vector field.

✓ chain rule

$$f(x^0, x^1, x^2, x^3) \longrightarrow \frac{d}{d\tau} f(x^0(\tau), \dots, x^3(\tau)) = \left(\frac{dx^\alpha}{\partial \tau}\right) \frac{\partial f}{\partial x^\alpha}$$

$$\boxed{\frac{Df}{D\tau} \stackrel{\text{def}}{=} \frac{\partial x^\alpha}{\partial \tau} \partial_\alpha f = u^\alpha \partial_\alpha f}$$

$$u^\alpha \rightarrow (\gamma, \gamma \vec{v}) \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$



$$\xrightarrow{NR} (1, \vec{v}) + O(v^2)$$

$$u^\alpha \partial_\alpha f = u^0 \partial_0 f + u^i \partial_i f \xrightarrow{NR} \partial_0 f + v_i \partial_i f$$

MCRF : (t, x, y, z)
(Momentarily Comoving RF of □)

Number Conservation

n = density $\frac{\Delta N}{\Delta V}$ in MCRF $\Rightarrow$ Lorentz Invariant

RF $\boxed{K}$: Let J^0 = density $\frac{\Delta N}{\Delta V}$

4-current (flux)

$$J^\alpha \longrightarrow \begin{pmatrix} J^0 \\ j_x \\ j_y \\ j_z \end{pmatrix} \begin{matrix} \text{density} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right] \text{flux} \end{matrix}$$

In MCRF (t,x,y,z) : $J^\alpha \longrightarrow \begin{pmatrix} n \\ 0 \\ 0 \\ 0 \end{pmatrix}$

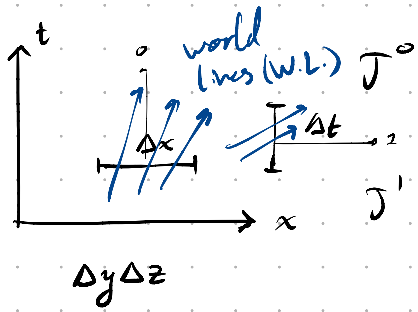
$$u^\alpha \to \begin{pmatrix} \gamma \\ \gamma \vec{v} \end{pmatrix} \xrightarrow{MCRF} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In MCRF $\boxed{J^\alpha = n u^\alpha}$ [True in one RF ✓ ; Tensor ✓]

$\Rightarrow$ True in any RF

Density & Flux

$\underbrace{\text{Density \& Flux}}$ $\downarrow$ $\boxed{\text{4-vector}}$



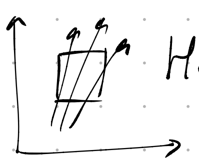
$J^0 = \text{density} = \dfrac{\#WL}{\Delta x \Delta y \Delta z}$, # = "Number of..."

$J^1 = \boxed{\text{Flux}_x} = \dfrac{\#WL}{\Delta y \Delta z \Delta t}$

Number Conservation Law

$$0 = \partial_0 J^0 + \partial_i J^i$$

$$\Rightarrow \boxed{0 = \partial_\alpha J^\alpha} \quad \text{or} \quad 0 = \partial_\alpha (n u^\alpha) \quad \text{4D Divergence}$$



Hypervolume $\Delta x \Delta y \Delta z \Delta t$

#WL entering = #WL exiting

Energy-Momentum conservation

Particle $p^\alpha \to \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$ ← Energy,] momentum

compare $J^\beta \to \begin{pmatrix} J_0 \\ J_1 \\ J_2 \\ J_3 \end{pmatrix}$ density,] ← flux

Energy-momentum Tensor

$T^{\alpha\beta}$ ← density/flux; Energy/momentum

T^{00} = Energy Density = $\dfrac{\Delta E}{\Delta V}$

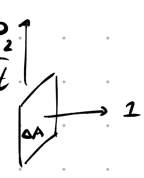
T^{i0} = Momentum Density

T^{0i} = Energy Flux = $\dfrac{\Delta E}{\Delta t \, \Delta A \perp_j}$

T^{ij} = Momentum Flux = $\dfrac{\Delta P}{\Delta t \, \Delta A \perp_j}$

"Stress Tensor"

$F_2 = \dfrac{\Delta P_2}{\Delta t}$



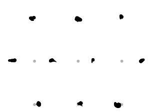
Formula of the day

$$T^{\alpha\beta} = (\rho - p)\, u^\alpha u^\beta + p\,\eta^{\alpha\beta}$$

$\boxed{\text{MCRF}}$

Assume Fluid made of particles all at rest. (Cosmology: "Dust")

MCRF $\quad |\text{velocity}| \ll 1$ (Macro + Micro)



✓ no momentum $\rightarrow$ $T^{00} = mn$, m = mass of particles (rest)

✓ no flux $\qquad T^{0i} = T^{i0} = T^{ij} = 0$

$$T^{\alpha\beta} \longrightarrow \begin{pmatrix} mn & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad u^\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^\alpha u^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

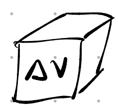
$\Rightarrow \boxed{T^{\alpha\beta} = \rho\, u^\alpha u^\beta}$, $\rho = \dfrac{\Delta\text{mass}}{\Delta V}$ in MCRF

$\Rightarrow$ FOTD when $p = 0$

ρ Lorentz Invariant

$\boxed{K} \quad u^\alpha = \begin{pmatrix} \gamma \\ \gamma\vec{v} \end{pmatrix}$

$$\left\{ \begin{aligned} T^{00} &= \rho\gamma^2 = mn\gamma^2 \\ T^{0i} &= \rho\gamma^2 v_i \\ T^{i0} &= \rho\gamma^2 v_i \\ T^{ij} &= \rho\gamma^2 v_i v_j \end{aligned} \right.$$



$\bigcirc\!\!\!\!K$ $\qquad J^\alpha = n u^\alpha$

$$T^{00}\Delta V = \Delta V\,(n\gamma)(m\gamma) = \underbrace{\Delta V\, J^0}_{\Delta N} m\gamma$$

$$= \sum_k m_k \gamma_k = \sum_k E_k$$

$$\left.\begin{aligned} T^{i0}\Delta V &= \rho\gamma^2 v_i \Delta V = (\rho n \Delta V) m\gamma v_i = \sum_k^{\Delta N} P_k \\ T^{0i}\Delta V &= \rho\gamma^2 v_i \Delta V = \sum_{\#} (m_\#\gamma_\#)\, v_{\# i} = \sum_{\#} E_\# v_{\# i} \end{aligned}\right\} \Rightarrow T^{\alpha\beta} = T^{\beta\alpha}$$

$$T^{ij}\Delta V = \sum_{\#} P_{\# i} v_{\# j}$$

Lec 9 - 2/13

Einstein's Equations

$$G_{\alpha\beta} = 8\pi G_N T_{\alpha\beta} \to \text{Energy-Momentum}$$

$G_{\alpha\beta}$: Einstein Tensor, related to curvature

G_N: Newton's Const

S.R.

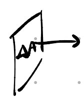
$T^{\alpha\beta}$ = Energy-Momentum tensor (Fluid)

MCRF (t, x, y, z)

ρ = mass density $= \frac{\Delta m}{\Delta V}$ $\qquad T^{00}$ = Energy density $= \frac{\Delta E}{\Delta V}$

n = number density $= \frac{\Delta N}{\Delta V}$ $\qquad T^{i0}$ = momentum $(i = 1,2,3)$ density $= \frac{\Delta P_i}{\Delta V}$

$u^\alpha = (\gamma, \gamma\vec{v})$ $\qquad T^{0j}$ = flux of energy $= \frac{\Delta E}{\Delta A_j \Delta t}$ $(\perp_j)$



T^{ij} = flux of momentum $= \frac{\Delta P_i}{\Delta A_j \Delta t} = \frac{\Delta F_i}{\Delta A_j}$

$$\text{MCRF } T^{\alpha\beta} \longrightarrow \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(particles all with same velocity)

$$\boxed{T^{\alpha\beta} = \rho u^\alpha u^\beta}, \quad u^\alpha \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

KI

$$u^\alpha \to \begin{pmatrix} \gamma \\ \gamma v_1 \\ \gamma v_2 \\ \gamma v_3 \end{pmatrix} \qquad n u^\alpha \to \begin{pmatrix} n\gamma \\ n\gamma v_1 \\ n\gamma v_2 \\ n\gamma v_3 \end{pmatrix}, \quad \rho = mn, \ m = \text{rest mass of particle}$$

$T^{00} = mn\, u^0 u^0 = mn\gamma^2$

$T^{0i} = T^{i0} = mn\gamma^2 v_i$

$T^{ij} = mn\gamma^2 v^i v_j$

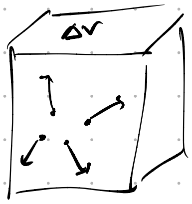
$$T^{00}\Delta V = mn\gamma^2 \Delta V = m\gamma \underbrace{(n\gamma\Delta V)}_{\Delta N \text{ in } \Delta V} = m\gamma\Delta N$$

$$T^{i0}\Delta V = (m\gamma v_i)(n\gamma\Delta V) = m\gamma v_i \cdot \Delta N$$

$$T^{0i}\Delta V = (m\gamma)\cdot(v_i)\Delta N = E \cdot v_i \cdot \Delta N$$

$$T^{ij}\Delta V = (m\gamma v_i)(v^j)\Delta N = P_i v_j \Delta N$$

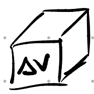
Now generalize to particles which move in many directions



MCRF

$$u^\alpha \to \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad J^\alpha \to \begin{pmatrix}n\\0\\0\\0\end{pmatrix}$$

KI



$$T^{00}\Delta V = \sum_{\#\text{ in }\Delta V} E_\# = \sum_\# m\gamma_\#$$

$$T^{0i}\Delta V = T^{i0}\Delta V = \sum_\# m\gamma_\# v_{\# i}$$

$$T^{ij}\Delta V = \sum_\# m\gamma_\# v_{\# i} v_{\# j}$$

Assumption Non-Interacting Particles

Perfect Fluid

these depend on (t, x, y, z)

$\boxed{\text{MCRF}}$ Definition $T^{\alpha\beta} \longrightarrow \begin{pmatrix}\rho & & & \\ & p & & \\ & & p & \\ & & & p\end{pmatrix} \longrightarrow$ No viscosity

p: "pressure"

Dust: $|p| \ll \rho \Rightarrow |v_i| \ll 1$ ($c \neq 1$ units: $|p| \ll \rho c^2$)

$$T^{00}\Delta V = \sum_{\#\text{ in }\Delta V} E_\# = \sum_\# m\gamma_\# \longrightarrow \rho$$

$$T^{ij}\Delta V = \sum_\# m\gamma_\# v_{\# i} v_{\# j} \longrightarrow p \quad (T^{ii} \text{ in MCRF})$$

① Tensor equation for $T^{\alpha\beta}$

② Conservation of E, P ← "$F = ma$"

MCRF

$$\rho u^\alpha u^\beta \longrightarrow \begin{pmatrix}\rho & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \qquad p\eta^{\alpha\beta} = \begin{pmatrix}-p & 0 & 0 & 0\\ 0 & p & 0 & 0\\ 0 & 0 & p & 0\\ 0 & 0 & 0 & p\end{pmatrix}$$

Perfect Fluid: $\boxed{T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p\eta^{\alpha\beta}}$

Conservation Law

$$J^\alpha \to \begin{pmatrix}J^0\\J^1\\J^2\\J^3\end{pmatrix} \qquad \partial_\alpha J^\alpha = 0 \Rightarrow \partial_\alpha T^{\beta\alpha} = 0 \xrightarrow{\text{sym}} \boxed{\partial_\alpha T^{\alpha\beta} = 0}$$

Fake: MCRF

using $T = \begin{pmatrix} \rho & \\ & p \end{pmatrix}$

$\cancel{\partial_0 \rho = 0}$

$\cancel{\partial_1 p = 0}$

$\cancel{\partial_2 p = 0}$

$\cancel{\partial_3 p = 0}$

MCRF at t $\neq$ MCRF at $t+\Delta t$, so $\rho \neq \rho'$, $p \neq p'$

Soln: momentum

$$\beta = i \; : \quad 0 = \partial_\alpha T^{\alpha i} = \partial_0 T^{0i} + \sum_{j=1}^{3} \partial_j T^{ji} = \partial_\alpha\left[(\rho+p) u^\alpha u^i\right] + \partial_\alpha\left[p \eta^{\alpha i}\right]$$

$$\partial_\alpha\left[p\eta^{\alpha i}\right] = \partial_i p$$

$$\partial_\alpha\left[(\rho+p)u^i u^\alpha\right] = \partial_{\alpha \to 0}\left[(\rho+p)u^i\right] u^{\alpha \to 0} + (\rho+p)\cancel{u^i}^{\,0}(\partial_\alpha u^\alpha) \xrightarrow{\text{MCRF}} \partial_0\left[(\rho+p)u^i\right]$$

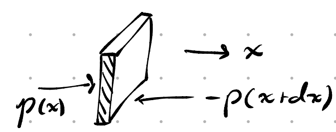
Keep $O(\partial_\alpha u^i)$ 1st order in velocity

$u^i \xrightarrow{\text{MCRF}} 0 \qquad \partial_\alpha u^0 = \partial_\alpha \gamma = \partial_\alpha\left(\frac{1}{\sqrt{1-v^2}}\right) = \frac{v\,\partial_\alpha v}{\sqrt{1-v^2}^{\,3}} \xrightarrow{\text{MCRF}} 0$

$\partial_\alpha u^i \xrightarrow{\text{MCRF}} ?$

$u^0 \xrightarrow{\text{MCRF}} 1$

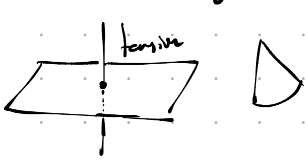
$$\boxed{0 = \partial_0\left[(\rho+p)u^i\right] + \partial_i p \;\Rightarrow\; \frac{d}{dt}\left[(\rho+p)\vec{v}\right] = -\vec{\nabla} p \;\Rightarrow\; (\rho+p)\vec{a} = -\vec{\nabla} p}$$



not just mc^2, since $\rho = \sum_\# E_\#$

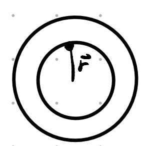
Explanation for $\rho + p$

Cosmic String



$\frac{GM(r)}{r^2}$

or $\quad (\rho + p)\vec{a} = -\vec{\nabla} p$



as p increases, $\vec{a}$ increases $\Rightarrow$ p cannot balance $\Rightarrow$ supernova

Energy $\beta = 0$: $0 = \partial_0 T^{00} + \partial_i T^{i0} = \partial_\alpha\left[(\rho+p)u^\alpha u^0\right] + \partial_i(p\eta^{i0})$

$0 = \dot{\rho} + \frac{\rho+p}{n}\partial_i(n u^i) \quad \Rightarrow \quad 0 = \dot\rho + \frac{\rho+p}{n}(-\dot n)$

$\partial_i(n u^i) = -\partial_0(n u^0) = -\dot n$

$\partial_0 u^0 \xrightarrow{\text{MCRF}} 0$

$\partial_i(p\eta^{i0}) = -\partial_0 p \quad (i=0)$

$\partial_\alpha\left[(\rho+p)u^0 u^\alpha\right]$

$= \partial_0(\rho+p) + \partial_i\left[\frac{(\rho+p)}{n} u^0 n u^i\right]$

$\xrightarrow{\text{MCRF}} \dot\rho + \dot p + \frac{\rho+p}{n} \cdot 1 \; \partial_i(n u^i)$

Lec 10 - 2/15

- Energy conservation for perfect fluid
- Accelerated observer

Perfect Fluid

MCRF $\quad T^{\alpha\beta} \longrightarrow \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$ $\quad \Delta V$ @ t, x, y, z

$u^\alpha \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, but $\partial_\alpha u^i \neq 0$

yet $\partial_\alpha u^0 = 0$

number conservation

$$\partial_\alpha (n u^\alpha) = 0$$

MCRF $\hookrightarrow \quad \partial_0 (n u^0) = -\partial_i (n u^i)$

$$\parallel$$

$$\partial_0 n = \dot{n}$$

Any K: $\quad T^{\alpha\beta} = (\rho + p) u^\alpha u^\beta + p \eta^{\alpha\beta}$

$\boxed{0 = \partial_\alpha T^{\alpha\beta}}$ $\quad E/p$ conservation

$u^\alpha \to (\gamma, \gamma \vec{v}) \qquad \beta = i = 1, 2, 3 \quad \Rightarrow \quad 0 = \partial_\alpha T^{\alpha\beta} \xrightarrow{MCRF} (\rho + p) \dot{\vec{v}} = -\vec{\nabla} p$

$\beta = 0 \qquad \Delta V \to dV \qquad\qquad \Delta V \ , \ \Delta m_{\text{inertial}} = (\rho + p) \Delta V$

Claim $\quad 0 = \partial_\alpha T^{\alpha\beta} \xrightarrow{MCRF} \text{.... Algebra ...} \to dE = -p\, dV$

$$0 = \partial_\alpha [(\rho + p) u^\alpha u^0] + \partial_\alpha [p \eta^{\alpha 0}]$$

$$\partial_\alpha u^0 = 0 \ , \ u^0 = 1$$

MCRF @ t_0, x_0, y_0, z_0 origin

$$\Rightarrow \partial_\alpha [(\rho + p) u^\alpha] - \partial_0 p = \partial_0 (\rho + p) + \partial_i [(\rho + p) u^i] - \partial_0 p$$

$\partial_i (n u^i) = -\dot{n}$

$$= \dot{\rho} + \partial_i \left[\frac{(\rho + p)}{n} \cdot n u^i \right] \Rightarrow 0 = \dot{\rho} + \left(\frac{\rho + p}{n} \right) \partial_i (n u^i) = \dot{\rho} - \left(\frac{\rho + p}{n} \right) \dot{n}$$

$E = mc^2$

$\Rightarrow \boxed{\dot{\rho} - \frac{\rho}{n} \dot{n} = \frac{p}{n} \dot{n}} \qquad dE = -p\, dV$ closed system, $\rho = \frac{\Delta E}{\Delta V}$, $n = \frac{\Delta N}{\Delta V}$

$$dE = -p\,dV \Rightarrow \underbrace{d}_{\text{over time}}\left(\frac{E}{N}\right) = -p\,d\left(\frac{V}{N}\right) \quad (\text{physical reinterpretation})$$

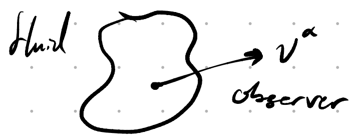
Math

$$\frac{V}{N} = \frac{1}{n}, \quad \frac{\rho}{n} = \frac{E/V}{N/V} = \frac{E}{N}$$

$$d \longrightarrow \frac{d}{dt} : dE = -p\,dV \Rightarrow \frac{d}{dt}\left(\frac{\rho}{n}\right) = -p\frac{d}{dt}\left(\frac{1}{n}\right) \Rightarrow \frac{\dot\rho}{n} - \frac{\rho}{n^2}\dot n = \frac{p}{n^2}\dot n \Rightarrow \dot\rho - \frac{\rho}{n}\dot n = \frac{p}{n}\dot n \;\checkmark$$

Thus, $dE = -p\,dV \longleftarrow \partial_\alpha T^{\alpha 0} = 0$.

General Observations



$$T_{\alpha\beta} v^\alpha v^\beta \xrightarrow{MCRF} \rho$$

observer $\longrightarrow v^\alpha = \dfrac{dx^\alpha}{d\tau}$

$\xrightarrow{K}$ Energy density

If K has 4-velocity v^α

Condition

$$\boxed{0 \le T_{\alpha\beta} u^\alpha u^\beta}$$ "Weak Energy Condition"

{for any timelike u^α

Aside:
Not true for zero-point energy in QM.

Why:
"4-momentum of ΔV fluid element in K

$$-T^{\alpha\beta} v_\beta \xrightarrow{K} \begin{pmatrix} \Delta E/\Delta V \\ \Delta\vec{p}/\Delta V \end{pmatrix}$$

$v_0 = -1$

require $-T^{\alpha\beta} v_\beta$ is time like 4-vector whenever v^α is time like.

"For every timelike v^α with $v^0 > 0$,
$W^\alpha = -T^{\alpha\beta} v_\beta$ is also timelike (or 0) with $W^0 \ge 0$."

"Dominant Energy Condition"

Perfect fluid: $w = \dfrac{p}{\rho}$

Examples

① Dust $w = 0$

② A gas of photons

$$\rho\Delta V = T^{00}\Delta V = \sum_{\#} E, \quad T^{ij}\Delta V = \sum_{\#} E\, v_i v_j$$

$$\Rightarrow 3p\Delta V = \sum T^{ii}\Delta V = \sum_{\#} E \underbrace{v^2}_{1 \text{ for photon}}$$

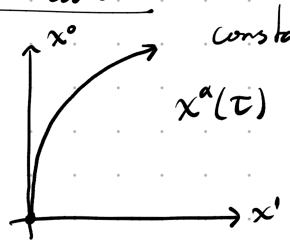
$w = \frac{1}{3}$,

$$v_{sound} = \sqrt{\left(\frac{dp}{d\rho}\right)_{adiabatic}} = \sqrt{w} = \frac{1}{\sqrt{3}}c$$

③ $w = -1$, $p = -\rho \rightarrow$ cosmological constant / Dark Energy

Acceleration

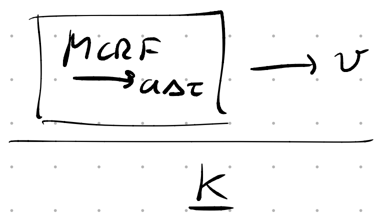
constant acceleration



1+1 dim.

a = constant in MCRF

$\tau \longrightarrow \tau + \Delta\tau$, $u^\alpha : 0 \to a\Delta\tau$



In K:

proper time	velocity
τ	$v(\tau)$
$\tau + \Delta\tau$	$\dfrac{a\Delta\tau + v}{1 + va\Delta\tau}$

$(c = 1)$

$$\Delta v = \frac{a\Delta\tau + v}{1 + va\Delta\tau} - v \underset{\text{taylor}}{=} a\Delta\tau + v(\cancel{1} - va\Delta\tau) - \cancel{v} = a(1-v^2)\Delta\tau$$

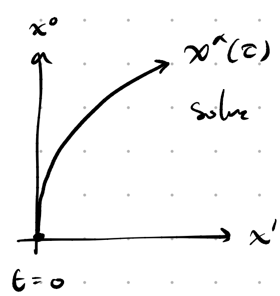
$$\frac{dv}{1-v^2} = a\,d\tau \Rightarrow \int \frac{dv}{1-v^2} = \tanh^{-1}(v) = a\tau \Rightarrow v = \tanh(a\tau)$$

$\longrightarrow$ rapidity $= a\tau$

$\gamma = \cosh(a\tau)$

$\gamma v = \sinh(a\tau)$

$u^\alpha \longrightarrow (u^0, u^1) = (\cosh(a\tau), \sinh(a\tau))$



solve $\dfrac{dx^\alpha}{d\tau} = u^\alpha$

$$x^0 = \int \cosh a\tau \, d\tau = \frac{1}{a}\sinh a\tau$$

$$x^1 = \int \sinh a\tau \, d\tau = \frac{1}{a}\cosh a\tau - \frac{1}{a}$$

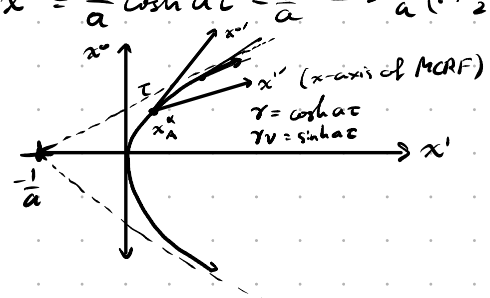
$\longleftarrow$ $(0,0)\ (\tau = 0)$

Accelerated Observer

$$x^0 = \frac{1}{a}\sinh a\tau \xrightarrow{a \text{ small}} \tau$$

$$x^1 = \frac{1}{a}\cosh a\tau - \frac{1}{a} \longrightarrow \frac{1}{a}\left(1 + \frac{1}{2}(a\tau)^2 + \cdots\right) - \frac{1}{a} = \frac{1}{2}a\tau^2 + \cdots$$

NR limit ✓



Coordinate system of acc. obs

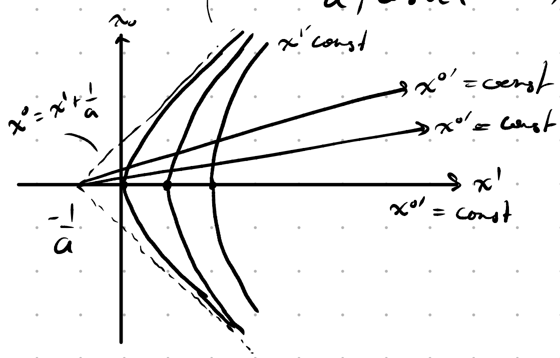
$\tau = x^{0'}$	$x^{1'}$

$$x^0 = \underbrace{\cancel{\gamma\Delta x^{0'}}}_{0} + \gamma v x^{1'} + x^0_A \Rightarrow x^0 = x^{1'}\sinh(a x^{0'}) + \frac{1}{a}\sinh(a x^{0'})$$

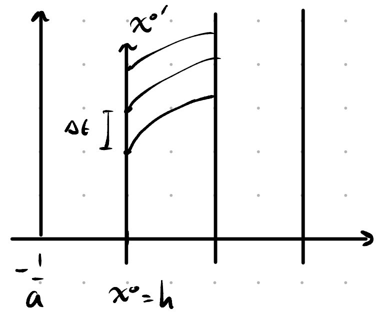
$$x^1 = \gamma x^{1'} + \underbrace{\cancel{\gamma v \Delta x^{0'}}}_{0} + x^1_A \Rightarrow x^1 = x^{1'}\cosh(a x^{0'}) + \frac{1}{a}\cosh(a x^{0'}) - \frac{1}{a}$$

$$x^0 = \left(x^{1'} + \frac{1}{a}\right)\sinh(a x^{0'})$$
$$x^1 = \left(x^{1'} + \frac{1}{a}\right)\cosh(a x^{0'}) - \frac{1}{a}$$

$K \rightarrow$ accelerated observer $(x^{0'}, x^{1'})$



acc obs

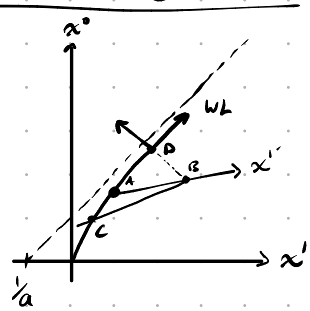


Lec 11 - 2/20

MT Feb 29. HW 1-4 → 2 pblms + 1
material up until today

Accelerated Observer

a = const acceleration in MCRF



$$x^0 = \frac{1}{a}\sinh a\tau$$

$$x^1 = \frac{1}{a}\cosh a\tau - \frac{1}{a}$$

If $\tau_{CA} = \tau_{AB}$, A, B simul.

RF (non-inertial) K'

$$x^0 = \left(\frac{1}{a} + x^{1'}\right)\sinh(a x^{0'})$$

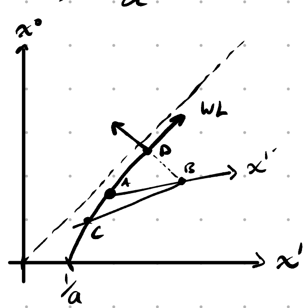
$$x^1 = \left(\frac{1}{a} + x^{1'}\right)\cosh(a x^{0'}) - \frac{1}{a}$$

Redefine origins

$$x^1 \to x^1 + \frac{1}{a}$$

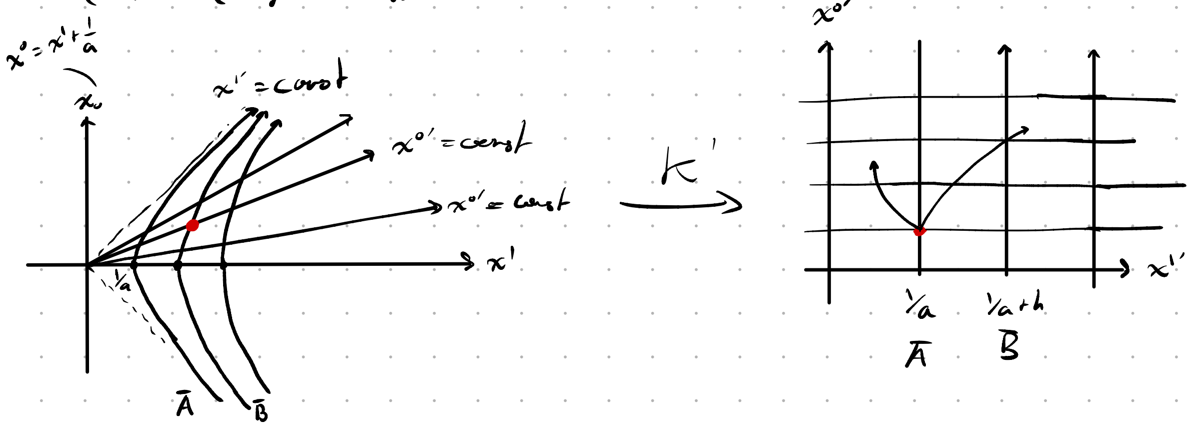
$$x^{1'} \to x^{1'} + \frac{1}{a}$$

$\Longrightarrow$



$$\begin{cases} x^0 = x^{1'}\sinh(a x^{0'}) \\ x^1 = x^{1'}\cosh(a x^{0'}) \end{cases}$$

$$(x^1)^2 - (x^0)^2 = \text{const}$$



Q Is hyperbola $x^{1'} = \frac{1}{a} + h$ also worldline of an observer with acceleration $= a$?

$$x^{1'} = \frac{1}{a} + h$$

$$x^0 = \left(\frac{1}{a} + h\right)\sinh(a x^{0'})$$

$$x^1 = \left(\frac{1}{a} + h\right)\cosh(a x^{0'})$$

Proper time for $\bar{B}$?

$$x^{0'} \to x^{0'} + dx^{0'}$$

$$d\tau^2 = -ds^2 = +(dx^0)^2 - (dx^1)^2$$

$$= \left(\tfrac{1}{a}+h\right)^2 \left\{\cosh^2(ax^{0'})\,[a\,dx^{0'}]^2 - \sinh^2(ax^{0'})\,[a\,dx^{0'}]^2\right\}$$

$$\Rightarrow d\tau^2 = \left(\tfrac{1}{a}+h\right)^2 a^2 (dx^{0'})^2 \left(\cancelto{1}{\cosh^2 - \sinh^2}\right)$$

$$\Rightarrow d\tau = (1+ah)\,dx^{0'}$$

$$\Rightarrow \tau = (1+ah)\,x^{0'} + \cancelto{0}{\text{const}}$$

Thus, formula for world line of $\bar{B}$ is $x^0 = \left(\frac{1}{a}+h\right)\sinh\left(\frac{a\tau}{1+ah}\right)$

$x^1 = \left(\frac{1}{a}+h\right)\cosh\left(\frac{a\tau}{1+ah}\right)$

<u>Concl.</u> observer at $a+\frac{1}{h}$ has

acc. $= \frac{a}{1+ah}$ or $\frac{1}{a}+h = \frac{1}{\text{acc}}$.

Time runs faster by $1+ah \xrightarrow{SI} 1 + \frac{ah}{c^2}$

<u>Ex</u>: $g = 9.8\,m/s^2 \longrightarrow \frac{ah}{c^2} = 1.1\times 10^{-11}\cdot\left(\frac{h}{100\,km}\right)$

Redshift

some constant

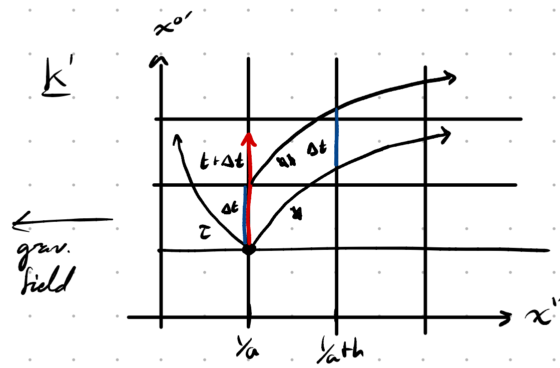
Light ray: $x^1 - x^0 = c$ (in K)

$$\Rightarrow x^{1'}\left[\cosh(ax^{0'}) - \sinh(ax^{0'})\right] = c$$

$$\Rightarrow x^{1'}e^{-ax^0} = c \Rightarrow x^{1'} = ce^{ax^{0'}}$$

$$\Rightarrow x^{0'} = \frac{1}{a}\log(x^{1'}/c)$$

To ⓛ: $x^{0'} = -\frac{1}{a}\log(x^{1'}/c)$



$$x^{0'} = \frac{1}{a}\log x^{1'} + \tilde{c}$$

t at $x^{1'} = \frac{1}{a} \Rightarrow$ * $x^{0'} = \frac{1}{a}\log x^{1'} + t - \frac{1}{a}\log\left(\frac{1}{a}\right)$

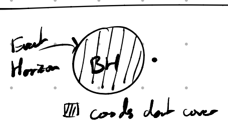
** $x^{0'} = \frac{1}{a}\log x^{1'} + t + \Delta t - \frac{1}{a}\log\left(\frac{1}{a}\right)$

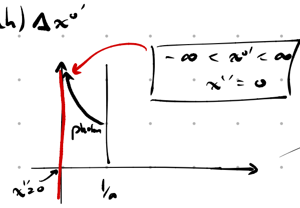
Thus, $\Delta x^{0'}$ agreed by $\bar{A}, \bar{B}$

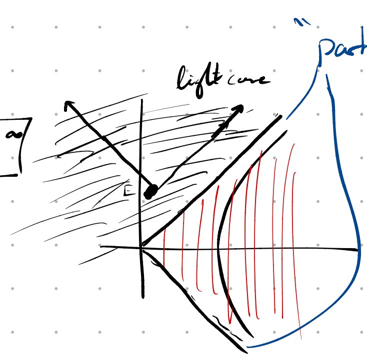
$\bar{A}$ $\Delta x^{0'} = \Delta$ prop time

$\bar{B}$ $\Delta\tau = (1+ah)\Delta x^{0'}$

Rindler Coordinates







Rindler Wedge is the only "visible" area of events

Spacetime Interval

$$ds^2 = -(dx^0)^2 + (dx^1)^2$$

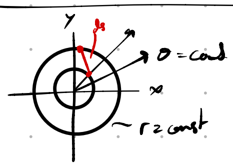
$$dx^0 = \sinh(ax^{0'})\, dx^{1'} + a x^{1'} \cosh(ax^{0'})\, dx^{0'}$$

$$dx^1 = \cosh(ax^{0'})\, dx^{1'} + a x^{1'} \sinh(ax^{0'})\, dx^{0'}$$

$$ds^2 = \text{algebra} = \boxed{-(ax^{1'})^2 dx^{0'2} + (dx^{1'})^2}$$ "metric of Rindler space"

Aside: Hawking → Unruh, see BBR from event horizon

Polar Coordinates (analogy)



$y = r\sin\theta$

$x = r\cos\theta$

$$ds^2 = dx^2 + dy^2 = r^2 d\theta^2 + dr^2$$

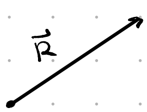
Analogy: $x^{0'} \to \theta$ $\sinh \to \sin$

$x^{1'} \to r$ $\cosh \to \cos$

General Relativity

Curvilinear Coordinates

Euclidean Space: $X^1, \dots, X^n$



$(x^1, \dots, x^n)$

$x^\mu,\ \mu = 1, \dots, n$

Ex: $n = 2$

$x^1 = r$ $\to \vec R = \begin{pmatrix} x_1 \cos x_2 \\ x_1 \sin x_2 \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

$x^2 = \theta$

Vectors

$\vec V = V^\mu \vec e_\mu$ $\vec e_1, \dots, \vec e_n$ a basis

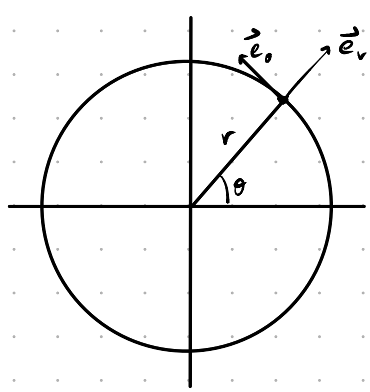
$$\vec e_\mu \equiv \partial_\mu \vec R = \frac{\partial \vec R}{\partial x^\mu}$$

Ex

$$\vec e_1 = \frac{\partial R}{\partial x_1} = \frac{\partial R}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$\vec e_1 = \vec e_r$

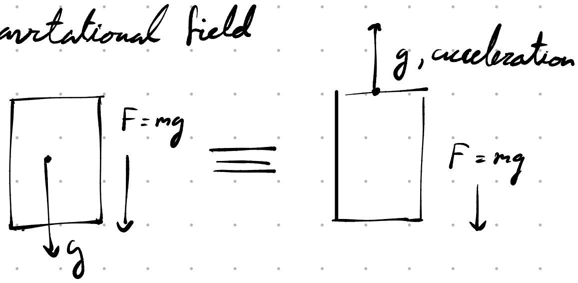
$$\vec e_2 = \frac{\partial R}{\partial x_2} = \frac{\partial R}{\partial \theta} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}$$



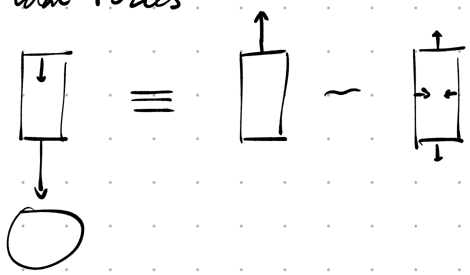
Lec 12 - 2/22

Einstein Equivalence Principle for Infinitesimal boxes

gravitational field



Tidal Forces

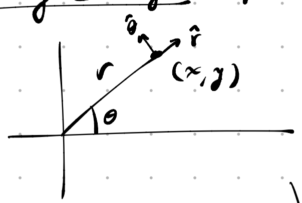


→ curvature of spacetime

We need to generalize for non-inertial reference frames:

- Vectors
- Tensors
- Interval
- Lowering / Raising
- ∂_μ

Running Example: Polar coordinates in 2D (x,y) ← Euclidean

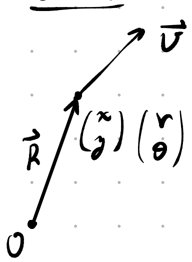


$$x^1 = x = r\cos\theta \qquad \longleftrightarrow \qquad x^{1'} = r = \sqrt{x^2+y^2}$$
$$x^2 \quad y = r\sin\theta \qquad\qquad x^{2'} = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Vector components: $\mu = 1, 2$ (x, y) $\quad \mu' = 1', 2'$ (r, θ)

$V^\mu : V^x, V^y \qquad V^{\mu'} : V^r, V^\theta$

Basis



Basis 1: $\vec{e}_\mu \equiv \dfrac{\partial \vec{R}}{\partial x^\mu} \equiv \partial_\mu \vec{R}$

x, y

$$\vec{R} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \vec{e}_x = \frac{\partial \vec{R}}{\partial x} = (1,0), \quad \vec{e}_y = \frac{\partial \vec{R}}{\partial y} = (0,1)$$

r, θ

$$\vec{R} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \quad \vec{e}_r = \frac{\partial \vec{R}}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \vec{e}_\theta = \frac{\partial \vec{R}}{\partial \theta} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}$$

(Not used in GR)

unit vectors: $\hat{r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, $\hat{\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ → $\vec{e}_r = \hat{r}$, $\vec{e}_\theta = r\hat{\theta}$

$$\vec{V} = V^\mu \vec{e}_\mu = V^{\mu'} \vec{e}_{\mu'}$$

V^μ = components of $\vec{V}$ $(\mu = 1, 2)$

r, θ $\quad V^{\mu'} \rightarrow \vec{V} = V^r \vec{e}_r + V^\theta \vec{e}_\theta = \begin{pmatrix} V^r\cos\theta - V^\theta r\sin\theta \\ V^r\sin\theta + V^\theta r\cos\theta \end{pmatrix}$

contravariant components of $\vec{V}$

Basis 2 $\quad \vec{\omega}^\mu = \vec{\nabla} x^\mu$

x, y

$$\vec{\omega}^1 = \vec{\nabla} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\vec{\omega}^2 = \vec{\nabla} y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

r, θ

$$\vec{\omega}^r = \vec{\nabla} r = \vec{\nabla}\sqrt{x^2+y^2} = \begin{pmatrix} \frac{x}{\sqrt{\,}} \\ \frac{y}{\sqrt{\,}} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \hat{r}$$

$$\vec{\omega}^\theta = \vec{\nabla}\theta = \vec{\nabla}\tan^{-1}\left(\frac{y}{x}\right) = \begin{pmatrix} -\frac{\sin\theta}{r} \\ \frac{\cos\theta}{r} \end{pmatrix} = -\frac{\sin\theta}{r}\vec{\omega}_x + \frac{\cos\theta}{r}\vec{\omega}_y$$
$$= \frac{1}{r}\hat{\theta}$$

$\vec{V} = V_\mu \vec{\omega}^\mu$, $\quad V_\mu$ covariant components of $\vec{V}$

Change of coord system.

$$X^\mu \longrightarrow X^{\mu'} \qquad x^{\mu'} = f^{\mu'}(x^1, x^2)$$

$$V^\mu \longrightarrow V^{\mu'} = \,? \quad \Rightarrow \quad \text{Jacobian (Matrix)}$$

$$\Lambda^{\mu'}_{\ \mu} = \frac{\partial x^{\mu'}}{\partial x^\mu}$$

$$\vec{e}_\mu = \frac{\partial \vec{R}}{\partial x^\mu} = \left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right)\frac{\partial \vec{R}}{\partial x^{\mu'}} = \Lambda^{\mu'}_{\ \mu}\,\vec{e}_{\mu'}$$

$$\vec{V} = V^\mu \vec{e}_\mu = V^\mu \Lambda^{\mu'}_{\ \mu}\,\vec{e}_{\mu'} \Rightarrow \boxed{V^{\mu'} = \Lambda^{\mu'}_{\ \mu} V^\mu}$$ Transformation of contravariant vector

Example (switched)

$$\underbrace{r, \theta}_{\mu} \qquad \underbrace{x, y}_{\mu'}$$

V^r, V^θ given $\qquad \begin{matrix} V^x \\ V^y \end{matrix} = \,?$

$$\Lambda^{\mu'}_{\ \mu} \rightarrow \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} C_\theta & -r S_\theta \\ r S_\theta & r C_\theta \end{pmatrix}$$

$$V^x = \Lambda^x_{\ r} V^r + \Lambda^x_{\ \theta} V^\theta = C_\theta V^r - r S_\theta V^\theta$$

$$\vec{V} = \begin{pmatrix} C_\theta V^r - r S_\theta V^\theta \\ \dots \end{pmatrix}$$

Covariant Vectors

V_μ given, $\; V_{\mu'} = \,?$

$$\vec{\omega}^\mu = \vec{\nabla} x^\mu. \quad \vec{R} = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}$$

$$(\vec{\omega}^\mu)_I = \frac{\partial x^\mu}{\partial X^I} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\frac{\partial x^{\mu'}}{\partial X^I} = \Lambda^\mu_{\ \mu'}\,(\vec{\omega}^{\mu'}) \Rightarrow \boxed{\vec{\omega}^\mu = \Lambda^\mu_{\ \mu'}\,\vec{\omega}^{\mu'}}$$

cartesian

$$(\vec{\omega}^{\mu'})_I = \frac{\partial x^{\mu'}}{\partial X^I}$$

$$\Rightarrow \vec{V} = V_\mu \vec{\omega}^\mu = V_\mu \Lambda^\mu_{\ \mu'}\,\vec{\omega}^{\mu'} \Rightarrow \boxed{V_{\mu'} = \Lambda^\mu_{\ \mu'} V_\mu}$$

Metric $g_{\mu\nu} \equiv \vec{e}_\mu \cdot \vec{e}_\nu$

Square of $\vec{V}$

$$\vec{V}\cdot\vec{V} = (V^\mu \vec{e}_\mu)\cdot(V^\nu \vec{e}_\nu) = g_{\mu\nu} V^\mu V^\nu$$

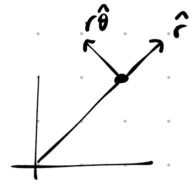
Example

$\vec{e}_r = \hat{r}$

$\vec{e}_\theta = r\hat{\theta}$

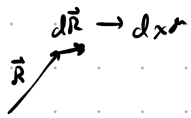
$$\begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$|\vec{V}|^2 = (V^r)^2 + r^2 (V^\theta)^2$$



Conveyed more compactly:

$$d\vec{R} = \begin{pmatrix} dx \\ dy \end{pmatrix} \qquad \boxed{|d\vec{R}|^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$$



$$d\vec{R} = \vec{e}_\mu dx^\mu$$

r,θ : $|d\vec{R}|^2 = dr^2 + r^2 d\theta^2$

$$\boxed{ds^2 = dr^2 + r^2 d\theta^2}$$

Consider $\vec{e}_\mu \cdot \vec{\omega}^\nu$ = algebra $= \delta^\nu_\mu$ $\rightarrow$ dual bases

$$\vec{V} = V^\mu \vec{e}_\mu = V_\mu \vec{\omega}^\mu \quad \boxed{V_\mu = \vec{V}\cdot\vec{e}_\mu}$$

Pf: $\vec{V}\cdot\vec{e}_\mu = V_\nu \vec{\omega}^\nu \cdot \vec{e}_\mu = V_\nu \delta^\nu{}_\mu = V_\mu$

Lowering Indices

$V^\mu \longrightarrow V_\mu = ?$: $V^\mu = \vec{V}\cdot\vec{e}_\mu = V^\nu \vec{e}_\nu \cdot \vec{e}_\mu = V^\nu g_{\nu\mu}$ $\Longrightarrow$ $\boxed{V^\mu = g_{\mu\nu} V^\nu}$

(r,θ) $V_r = g_{rr}V^r + g_{r\theta}V^\theta = V^r$, $V_\theta = g_{\theta r}V^r + g_{\theta\theta}V^\theta = r^2 V^\theta$

Inverse Metric

$g^{\mu\nu} = \vec{\omega}^\mu \cdot \vec{\omega}^\nu \longrightarrow (g^{-1})$ as a matrix (r,θ) : $g^{\mu\nu} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$

$$\boxed{V^\mu = g^{\mu\nu} V_\nu}$$

Tensors

$$T^\mu{}_\nu = u^\mu v_\nu, \quad \boxed{T^{\mu'}{}_{\nu'} = \Lambda^{\mu'}{}_\mu \Lambda^\nu{}_{\nu'} T^\mu{}_\nu}$$