Lec 1 - 1/16

MATH 141 Differential Topology

Instructor: Boris Mladenov
mladenov@berkeley.edu

Office Hours: 11-12/12:30, 1073 Evans

Grading: weekly homework, Tues, due next Tues before lecture.
Submissions on gradescope

Homework: 30%
Midterm : 20%
Final : 50%

Book: Differential Topology by Guillemin & Pollack

Also - Smooth Manifolds by Lee

- Topology from the differential viewpoint by Milnor

- Differential Topology by Hirsch

Background: - General Topology
- Multivariable Analysis
- Linear/Multi-linear Algebra
- Abstract Algebra

Smooth Manifolds, smooth maps

- $\mathbb{R}^n$
-  S^1
-  S^2
-  $T^2 \simeq S^1 \times S^1$
-  genus 3 surface
- GL_n - invertible $n \times n$ matrices
- Finite set of points in $\mathbb{R}^n$

Def A topological space X is locally Euclidean if

$\forall x \in X$, $\exists$ an open neighborhood $x \in \mathcal{U}$ and a homeomorphism

$\varphi: \mathcal{U} \to V \subseteq \mathbb{R}^n$, where V is open

Def A topological manifold is a second countable, Hausdorff, locally Euclidean topological space.

Def (smooth function)

Let $X \subseteq \mathbb{R}^n$ and consider a (cts) map $f : X \to \mathbb{R}^m$

We say that f is smooth at $x \in X$ if $\exists$ an open $x \in \mathcal{U} \subseteq \mathbb{R}^n$ and

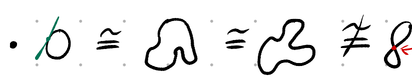
an extension $\tilde{f}$ $(\tilde{f}|_{\mathcal{U}\cap X} = f|_{\mathcal{U}\cap X})$ which is smooth, i.e. it has partial derivatives of all orders.

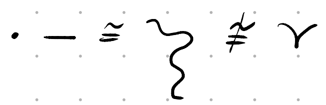
f is smooth on X if it is smooth at every $x \in X$.

Def (diffeomorphism) A map $f : \underset{\mathbb{R}^n}{\overset{\cap}{X}} \to \underset{\mathbb{R}^m}{\overset{\cap}{Y}}$ is a diffeomorphism if :

1) f is a bijection.
2) f is smooth.
3) f^{-1} is smooth.

If there is a diffeomorphism between X and Y, we say they are diffeomorphic and write $X \cong Y$

 ← issue, tangent space is $\mathbb{R}^2$



Difference between something admits a smooth structure and if its realization in $\mathbb{R}^n$ is smooth.

Def (Smooth (sub) manifold, 1^{st} attempt)

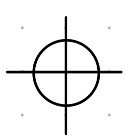
A subset $X \subseteq \mathbb{R}^n$ is a smooth (sub) manifold (of $\mathbb{R}^n$) of dimension k if

$\forall x \in X$, $\exists$ an open $x \in \tilde{\mathcal{U}} \subseteq \mathbb{R}^n$ and a diffeomorphism $\varphi : \tilde{\mathcal{U}} \cap X \xrightarrow{\cong} \overset{\text{open}}{V} \subseteq \mathbb{R}^k$.

The pair $(\tilde{\mathcal{U}}, \varphi)$ is called a smooth chart around $x \in X$.

Examples $\mathbb{R}^n$ trivially a smooth manifold. Simple smooth chart that covers $\mathbb{R}^n$: $(\mathbb{R}^n, id)$

$S^1 = \{x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$



4 subsets : $\mathcal{U}_{x>0} = \{x>0\} \cap S^1$
$\mathcal{U}_{x<0}, \mathcal{U}_{y>0}, \mathcal{U}_{y<0}$

$\varphi_{x>0} : \mathcal{U}_{x>0} \longrightarrow (-1,1) \subseteq \mathbb{R}$
$(x,y) \longmapsto y$

$\varphi_{x>0}^{-1} : (-1,1) \longrightarrow \mathcal{U}_{x>0}$
$y \longmapsto (\sqrt{1-y^2}, y)$ $\square$

Lec 2 - 1/18 Smooth Manifolds ~~and tangent spaces~~

Examples

1) $GL(n) \subseteq \mathbb{R}^{n^2}$ of invertible $n \times n$ matrices is a smooth manifold.

   $GL(n) = \det^{-1}(\mathbb{R}^{n^2} \setminus \{0\})$, where $\det : Mat(n) = \mathbb{R}^{n^2} \to \mathbb{R}$ is the determinant.

2) $O(n) \subseteq GL(n)$ - the space of orthogonal matrices, i.e., $A \in O(n) \Leftrightarrow AA^T = I$

   Let $so(n)$ be the space of $n \times n$ skew-symmetric matrices, i.e. $A \in so(n)$ iff $A = -A^T$

   $so(n)$ is diffeomorphic to $\mathbb{R}^{\binom{n}{2}}$

   Then the exponential $\exp : so(n) \to O(n)$ is a local diffeomorphism by the inverse function theorem

   $A \mapsto e^A$ nonzero $\Rightarrow$ locally invertible.

   This gives a chart around $I \in O(n)$, $e^0 = I$.

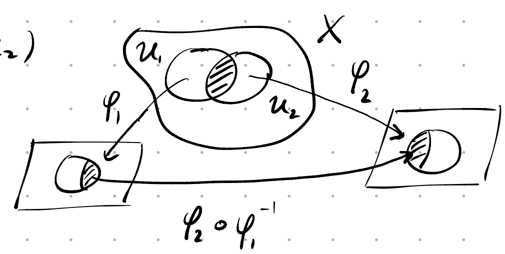
   Then for any other $A \in O(n)$, we can just translate using matrix multiplication

A second def of smooth manifolds

Def Let X be a topological manifold and let $(\mathcal{U}_1, \varphi_1)$ and $(\mathcal{U}_2, \varphi_2)$ be two charts.

The transition function from $(\mathcal{U}_1, \varphi_1)$ to $(\mathcal{U}_2, \varphi_2)$ is given by — ↳ have to be homeomorphisms

$$\varphi_{21} = \varphi_2 \circ \varphi_1^{-1}\big|_{\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2)} : \varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$$



Def: (Smooth Atlas)

Let X be a top manifold.

Suppose $(\mathcal{U}_\alpha, \varphi_\alpha)$, $\alpha \in I$ is an open cover of charts for X.

We say that collection $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is a smooth atlas

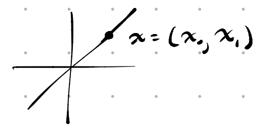
if the transition functions between any two charts are smooth.

Def: Two (smooth) atlases $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ are equivalent

if their union is a (smooth) atlas.

Exercise: Show the above relation between smooth atlases is an equivalence relation.

Def: A smooth manifold is a topological manifold together with an equivalence class of smooth atlases.



Example: Projective Space

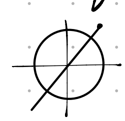
$\mathbb{R}P^n$ = set of lines in $\mathbb{R}^{n+1}$ through the origin.

representative of line

A point in $\mathbb{R}P^n$ can be represented by $[x_0, \dots, x_n]$, where at least one $x_i \neq 0$.

For any $\lambda \in \mathbb{R}\setminus\{0\}$: $[x_0, \dots, x_n] = [\lambda x_0, \dots, \lambda x_n]$

$\mathbb{R}P^n = \mathbb{R}^{n+1}\setminus\{0\} / \mathbb{R}^\times$ - This is given the quotient topology via the projection $\pi : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}P^n$

If we restrict to the sphere:  the smooth structure is easy. We do it by defn, however.

Let $\mathcal{U}_i = \underset{\text{open } \subset \mathbb{R}^{n+1}\setminus\{0\}}{\{x_i \neq 0\}} \cap \mathbb{R}P^n = \{[x_0, \dots, x_n] \in \mathbb{R}P^n : x_i \neq 0\}$

omitted

$$\varphi_i : \mathcal{U}_i \to \mathbb{R}^n, \quad [x_0, \dots, x_n] \longmapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right)$$

The inverse $(y_0, \dots, y_{n-1}) \longmapsto [y_0, \dots, \underset{i^{th} \text{ position}}{1}, \dots, y_{n-1}]$

Let's do $n = 1$: $(\mathcal{U}_0, \varphi_0)$, $(\mathcal{U}_1, \varphi_1)$

Then the transition function is $\mathbb{R}\setminus\{0\} \xrightarrow{\varphi_{10}} \mathbb{R}\setminus\{0\}$

$$\varphi_{10} = \varphi_1 \circ \varphi_0^{-1} : y \overset{\varphi_0^{-1}}{\longmapsto} [1, y] \overset{\varphi_1}{\longmapsto} \frac{1}{y} \quad \text{is smooth} \checkmark$$

In general, fix i, j say $i < j$. What does φ_{ji} look like?

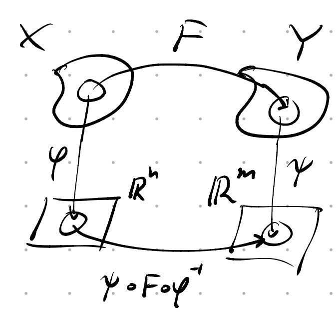
omitted

$$(y_0, \dots, y_{n-1}) \overset{\varphi_i^{-1}}{\longmapsto} [y_0, \dots, \underset{i^{th}}{1}, \dots, y_{n-1}] \overset{\varphi_j}{\longmapsto} \left(\frac{y_0}{y_j}, \dots, \frac{1}{y_j}, \dots, \widehat{\frac{y_j}{y_j}}, \dots, \frac{y_{n-1}}{y_j}\right)$$

This is smooth!

Def: Let X, Y be smooth manifolds and $F : X \to Y$ a cts map.

We say F is <u>smooth at $x \in X$</u> if there are charts $(\mathcal{U}, \varphi)$ around x and (V, ψ) around $F(x)$ such that the composition $\psi \circ F \circ \varphi^{-1} : \varphi(\mathcal{U}) \to \psi(F(\mathcal{U}))$ is smooth.



Lec 3 - 1/23 Tangent Spaces and Derivatives

Recall if $\mathcal{U}, V \subseteq \mathbb{R}^m$, then for a map $\varphi : \mathcal{U} \to V$, the derivative of φ at $x \in \mathcal{U}$

is the linear map s.t. $\varphi(y) = \varphi(x) + d\varphi_x (y - x) + \text{h.o.t.}$

$\quad$ ↖ matrix (Jacobian)

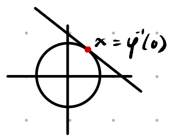
In particular, if φ is linear, then $d\varphi = \varphi$.

Let $X \subseteq \mathbb{R}^N$ be a smooth manifold, and let $(\mathcal{U}, \varphi)$ be a chart "around" $x \in X$, (assume $\varphi(x) = 0 \in \mathbb{R}^N$.)

Then, if $V = \varphi(\mathcal{U})$, we get a map $V \to \mathbb{R}^N$. This the linear approximation of φ^{-1}.

$$v \mapsto \varphi^{-1}(0) + (d\varphi^{-1})_0 \cdot v$$

Then, we define:



Def (Tangent Space)

The tangent space to X at $x \in X$ as the image of $d\varphi_0^{-1} : \overset{\varphi(\mathcal{U})}{\overset{\|}{V}} \to \mathbb{R}^N$.

$\quad$ ↳ over $\mathbb{R}^n$, where X is n-dim

This tangent space is denoted by $T_x X$. $\quad \varphi(\mathcal{U}) = V \subseteq \mathbb{R}^n$

Lemma: The tangent space is well-defined, i.e. independent of the choice $(\mathcal{U}, \varphi)$.

Pf: Let $(\mathcal{U}, \varphi)$ and (W, ψ) be two charts centered around $x \in X$.

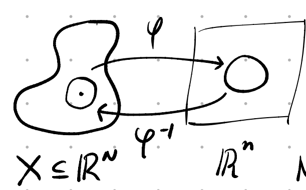
Then the transition function is $\varphi \circ \psi^{-1}$ (restricted to overlap), and we have.

$$\psi^{-1} = \varphi^{-1} \circ (\varphi \circ \psi^{-1}). \text{ Taking derivatives, we get } d\psi_0^{-1} = d(\varphi^{-1} \circ (\varphi \circ \psi^{-1}))_0 \overset{\text{chain rule}}{=} d\varphi_0^{-1} \circ d(\varphi \circ \psi^{-1})_0$$

composition / matrix mult

$\varphi \circ \psi^{-1}(0) = \varphi(x) = 0$

$$\begin{tikzcd}
\psi(W \cap \mathcal{U}) \arrow[r, "\varphi \circ \psi^{-1}"] \arrow[d, "\psi^{-1}"'] & \varphi(W \cap \mathcal{U}) \arrow[d, "\varphi^{-1}"] \\
W \cap \mathcal{U} \arrow[r, equal, "id"] & W \cap \mathcal{U}
\end{tikzcd}
\xrightarrow{d}
\begin{tikzcd}
\mathbb{R}^n \arrow[r, "d(\varphi \circ \psi^{-1})", "\cong"'] \arrow[d, "d\psi_0^{-1}"'] & \mathbb{R}^n \arrow[d, "d\varphi_0^{-1}"] \\
\mathbb{R}^N \arrow[r, equal] & \mathbb{R}^N
\end{tikzcd}$$

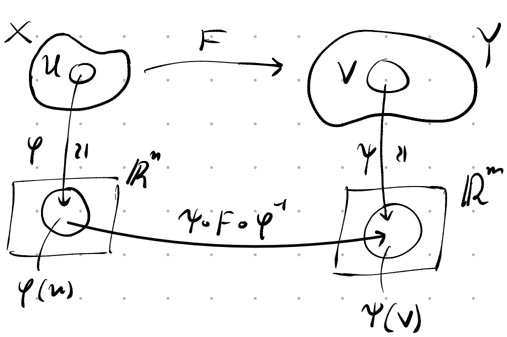


Note: $T_0 \mathbb{R}^n = \mathbb{R}^n$.

Let X, Y be manifolds and $F : X \to Y$ be a map.

For a point $x \in X$, we want to get a linear map $dF_x : T_x X \to T_{F(x)} Y$.

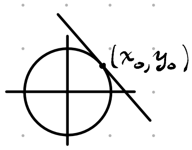
Let $(\mathcal{U}, \varphi), (V, \psi)$ be charts around $x \in X$ and $F(x) \in Y$.



Then
$$\begin{tikzcd}
T_x X \arrow[r, dashed, "dF_x"] \arrow[d, "(d\varphi_0^{-1})^{-1}"'] & T_{F(x)} Y \\
\mathbb{R}^n \arrow[r, "d(\psi \circ F \circ \varphi^{-1})_0"] & \mathbb{R}^m \arrow[u, "d\psi_0^{-1}"']
\end{tikzcd}$$

This is again independent of charts, which follows from the chain rule.

Example: $S^1 \subseteq \mathbb{R}^2$



$\mathcal{U} = S^1 \cap \{x > 0\}$ What is $T_{(x_0,y_0)} S^1$?

$\varphi : \mathcal{U} \to (-1,1)$ $\quad T_{(x_0,y_0)} S^1 \subseteq \mathbb{R}^2$

$(x,y) \mapsto y$

$\varphi^{-1} : y \mapsto (\sqrt{1-y^2}, y)$, $\quad y \in (-1,1)$

Then $d\varphi^{-1}_y = \begin{pmatrix} -\frac{y}{\sqrt{1-y^2}} \\ 1 \end{pmatrix} \Rightarrow$ the image is the corresponding tangent space $T_{(x,y)} S^1$.

In particular for (x_0, y_0): $\mathrm{Im} \begin{pmatrix} -\frac{y_0}{x_0} \\ 1 \end{pmatrix} = \mathrm{Im} \begin{pmatrix} -y_0 \\ x_0 \end{pmatrix}$ <u>the orthogonal to (x_0, y_0)</u>

$F_n : S^1 \to S^1$

$p \mapsto p^n$

$\exp : \mathbb{R} \to S^1$

$t \mapsto (\cos(t), \sin(t))$

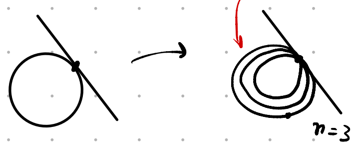
$$\begin{array}{ccc} S^1 & \xrightarrow{F_n} & S^1 \\ \uparrow \exp & & \uparrow \exp \\ \mathbb{R} & \xrightarrow[\cdot n]{} & \mathbb{R} \end{array} \quad \xrightarrow{d} \quad \begin{array}{ccc} TS^1 & \xrightarrow{dF} & TS^1 \\ \uparrow d\exp & & \uparrow d\exp \\ \mathbb{R} & \xrightarrow[\cdot n]{} & \mathbb{R} \end{array} \quad \begin{array}{ccc} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} & \xmapsto{n \cdot F_n} & \begin{pmatrix} -n \sin nt \\ n \cos nt \end{pmatrix} \\ \uparrow & & \uparrow \\ t & \longmapsto & nt \end{array}$$

← multiplication

→ multiplication by n:

$d\exp = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

3 times around circle



<u>Def</u>: A curve in X through $x \in X$ is a <u>smooth map</u> $\sigma : (-\varepsilon, \varepsilon) \to X$
$\sigma(0) = x$

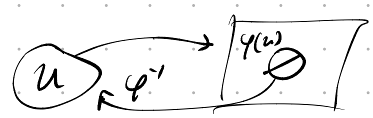
$X \subseteq \mathbb{R}^N$. For a chart $(\mathcal{U}, \varphi)$:

$$\begin{array}{ccc} T_x X & & v \\ d\varphi_x \downarrow \wr\Vert & & \downarrow \\ \mathbb{R}^n & & (d\varphi_x) v \end{array}$$

Fix a choice for $v \in T_x X$

There is a line in X through x

such that its velocity at 0 is v: $\sigma(0) = x$, $d\sigma_0 = v$



We have a line in $\mathbb{R}^n$ through 0 and v, then the pull-back under φ gives a curve in X with the required properties.

Lec 4 - 1/25

Last time

$X \subseteq \mathbb{R}^N \rightsquigarrow T_x X, \ \forall x \in X$

$F : X \to Y \rightsquigarrow T_x X \xrightarrow{dF_x} T_{F(x)} Y$

For a chart $(\mathcal{U}, \varphi)$, the image of $\mathcal{U}$ under φ is $\tilde{\mathcal{U}}$. $\varphi : \mathcal{U} \to \tilde{\mathcal{U}}$. Then if $F : X \to Y$, $\tilde{F} : \tilde{\mathcal{U}} \to \tilde{\mathcal{V}}$ $= \psi \circ F \circ \varphi^{-1}$

$(\mathcal{U}, \varphi)$ chart around x, then $T_x X = \mathrm{Im}\, d(\varphi^{-1})_0$, $\varphi : \mathcal{U} \to \tilde{\mathcal{U}} \subseteq \mathbb{R}^n$

$$\begin{array}{ccc} T_x X & \ni & v \\ \text{\rotatebox{90}{$\cong$}}\big\downarrow & & \big\downarrow \\ \mathbb{R}^n & & d\varphi_x(v) \end{array}$$

Let L be the line $t \mapsto t\, d\varphi_x(v)$. This is a line in $\mathbb{R}^n$.

After restricting the domain of $L : t \mapsto t\, d\varphi_x(v)$, we may assume $L : (-\varepsilon, \varepsilon) \to \tilde{\mathcal{U}}$, then we get a curve $\sigma = \varphi^{-1} \circ L : (-\varepsilon, \varepsilon) \to \mathcal{U}$ which satisfies 2 conditions:

$\sigma(0) = x$ - our choice of point

$d\sigma_0(1) = v$ - our choice of tangent vector

Def Let σ_1, σ_2 be two curves in X. We say they are <u>equivalent</u> if

$\sigma_1(0) = \sigma_2(0) \overbrace{= x}^{\text{can ignore}}$ and $d(\sigma_1)_0(1) = d(\sigma_2)_0(1)$.

The equivalence class is denoted $[c]$

Def Let X be a smooth manifold. Then the <u>tangent space</u> at $x \in X$ is the space of equivalence classes of curves in X such that $\sigma(0) = x$.

<u>Lemma</u>: The tangent space is a vector space of dimension $\dim X = n$

<u>Pf</u>: Let $(\mathcal{U}, \varphi)$ be a chart around $x \in X$. Then we define

$T_x X \to \mathbb{R}^n$

$[\sigma] \mapsto d(\varphi \circ \sigma)_0(1)$ This is a <u>bijection</u> by definition. (of $[\sigma]$)

This vector space structure is independent of $(\mathcal{U}, \varphi)$:

$$\begin{tikzcd} & T_x X \arrow[dl, "{(\mathcal{U},\varphi)}"'] \arrow[dr, "{(\mathcal{V},\psi)}"] & \\ \mathbb{R}^n \arrow[rr, "\cong", "d(\psi \circ \varphi^{-1})"'] & & \mathbb{R}^n \end{tikzcd}$$

$X \subseteq \mathbb{R}^N$

$F: X \to Y$

$$\begin{tikzcd}
v \arrow[d, mapsto] & T_xX \arrow[r, "dF_x"] \arrow[d, "d\varphi_x"', "\cong"] & T_{F(x)}Y \arrow[d, "d\psi_{F(x)}", "\cong"'] \\
d\varphi_x \cdot v \arrow[d, mapsto] & \mathbb{R}^n \arrow[r, "d\tilde{F}"] \arrow[d, "\cong"] & \mathbb{R}^m \arrow[d, "\cong"] \\
{[\sigma : t \mapsto \varphi^{-1}(t\, d\varphi_x v)]} & T_xX \arrow[r, "dF'_x"] & T_{F(x)}Y
\end{tikzcd}$$

$$d\tilde{F} \cdot d\varphi_x(v) = d(\psi \circ F \circ \varphi^{-1})_0 \cdot d\varphi_x(v)$$

$$[t \mapsto \psi^{-1}(t\, d(\psi \circ F \circ \varphi^{-1})_0 d\varphi_x(v))] = [\sigma_1] \qquad [\sigma] \mapsto [F \circ \sigma]$$

$$[\sigma] \longmapsto [F \circ \sigma]$$

$$\longrightarrow [t \mapsto F \circ \varphi^{-1}(t\, d\varphi_x(v))] = [\sigma_2]$$

Def Let X, Y be smooth manifolds, $F: X \to Y$ a smooth map. The derivative of F at x is the linear map $dF_x : T_xX \to T_{F(x)}Y$

$[\sigma] \mapsto [F \circ \sigma]$

Since $\varphi(x) = 0$, $\psi(F(x)) = 0$, $\sigma_1(0) = \sigma_2(0) = F(x)$. Then, we need $d(\sigma_1)_0(1) = d(\sigma_2)_0(1)$.

For σ_1, we get $d(\psi \circ \sigma_1)_0(1) = d\tilde{F}_0 \circ d\varphi_x(v)$.

For σ_2, we use (V, ψ) to get $d(\psi \circ \sigma_2) = d\left[(\psi \circ F \circ \varphi^{-1})_0 (t\, d\varphi_x(v))\right](1) = d\tilde{F}_0 \cdot d\varphi_x(v)$

Derivative of a curve is independent of the chart: take two charts $(\mathcal{U}_1, \varphi_1), (\mathcal{U}_2, \varphi_2)$ around x and assume $\sigma: (-\varepsilon, \varepsilon) \longrightarrow \mathcal{U}_1 \cap \mathcal{U}_2 \subseteq X$.

Then we have two ways to take the derivative:

$$\varphi_1 = \varphi_2 \circ \varphi_2^{-1} \circ \varphi_1 = \varphi_1 \circ \varphi_2^{-1} \circ \varphi_2$$

$$d(\varphi_1 \circ \sigma)_0(1) = d(\varphi_1 \circ \varphi_2^{-1} \circ \varphi_2 \circ \sigma)_0(1) = \underbrace{d(\varphi_1 \circ \varphi_2^{-1})_x}_{\text{linear isomorphism } \mathbb{R}^n \to \mathbb{R}^n} \circ d(\varphi_2 \circ \sigma)_0(1)$$

$$d(\varphi_2 \circ \sigma)_0(1) \qquad \mathbb{R} \to \mathbb{R}^n$$

$F_S^{-1} = F_{1/S}$ $\qquad (F \circ \varphi)^{-1} = \varphi^{-1} \circ F^{-1}$

$F: \mathcal{B} \xrightarrow{\sim} \mathcal{B}$ $\qquad F_S \circ \boxed{\varphi \circ \varphi^{-1}} \circ F^{-1}_{1/S}$

$0 \mapsto 0$ $\qquad \varphi \mapsto 0$

Def Let $F: X \to Y$ be a smooth map between manifolds. Then we say that F is

(1) an <u>immersion</u> at $x \in X$ if dF_x is injective.

(2) an <u>embedding</u> if F is an immersion at x $\forall x \in X$, and a homeomorphism onto its image.

(3) a <u>submersion</u> at $x \in X$ if dF_x is surjective.

<u>Examples</u>

1) $X \subseteq \mathbb{R}^N$ is a manifold, $i: X \hookrightarrow \mathbb{R}^N$ (inclusion) is an immersion. In fact, it is an embedding.

2) $8 \subseteq \mathbb{R}^2$ is an immersion. $F: \mathbb{R}$ or $S^1 \to \mathbb{R}^2$

3) $\prec \subseteq \mathbb{R}^2$ $\quad F: t \mapsto (t^2, t^3)$ $\quad dF_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Lec 5 - 1/30 immersions, submersions, embeddings

Inverse Function Thm (Calculus)

Let $F: \underset{\mathbb{R}^n}{\underset{\cap}{U}} \to \underset{\mathbb{R}^n}{\underset{\cap}{V}}$ be a smooth map, U, V open.

Assume that $dF_x : \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ is an isomorphism, where $x \in U$.

Then F is a local diffeomorphism around x, i.e. there is an open $x \in U_0 \subseteq U$ such that $F|_{U_0} : U_0 \to F(U_0) = V_0 \subseteq V$ is a diffeomorphism.

$$\begin{tikzcd} U_0 \arrow[r, "F|_{U_0}"] \arrow[d, "id_{U_0}"'] & V_0 \arrow[d, "(F|_{U_0})^{-1}"] \\ U_0 \arrow[r, "id_{U_0}"'] & U_0 \end{tikzcd}$$

Inverse Function Thm (Manifolds)

Let $F: X \to Y$ be a smooth map between smooth manifolds X and Y.

Assume that $dF_x : T_x X \to T_{F(x)} Y$ is an isomorphism.

Then F is a local diffeomorphism around x.

Pf: Let (U, φ) be a chart around x
(V, ψ) —————"————— $F(x)$.

Then we have

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ \tilde{U} \arrow[r, "\tilde{F}"] & \tilde{V} \\ \tilde{U}_0 \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\tilde{F}|_{\tilde{U}_0}"] & \tilde{V}_0 \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

diffeomorphism.

So dF_x is an isomorphism
$\Leftrightarrow$
$d\tilde{F}_0$ is an isomorphism

Then $F|_{\varphi^{-1}(\tilde{U}_0)}$ is a diffeomorphism

$F|_{\varphi^{-1}(\tilde{U}_0)} : \varphi^{-1}(\tilde{U}_0) \to \psi^{-1}(\tilde{V}_0)$ $\square$

Thm (Structure thm for immersions)

Let $F: X \to Y$ be a smooth map between manifolds X and Y and assume that F is an immersion at $x \in X$.
(i.e. dF_x is injective)

Then there are charts (U, φ) and (V, ψ) around x and $F(x)$ respectively, such that $\tilde{F}: \underset{\mathbb{R}^n}{\underset{\cap}{\tilde{U}}} \to \underset{\mathbb{R}^m}{\underset{\cap}{\tilde{V}}}$ is the map $(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n, 0, \dots, 0)$.

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ \tilde{U} \arrow[r, "\tilde{F}"'] & \tilde{V} \end{tikzcd}$$

Pf: Since dF_x is injective, $d\tilde{F}_0$ is injective. So we can assume $d\tilde{F}_0 : \mathbb{R}^n \to \mathbb{R}^m$ is of the form $\begin{pmatrix} Id_n \\ 0 \end{pmatrix}$.
(by changing basis)

Let
$$\begin{aligned} \Phi : \tilde{\mathcal{U}} \times \mathbb{R}^{m-n} &\to \tilde{V} \\ (x, z) &\mapsto \tilde{F}(x) + (0, z) \end{aligned}$$

Then $d\Phi_0$ is given by $\begin{pmatrix} id_n & 0 \\ 0 & id_{m-n} \end{pmatrix}$, so Φ is a diffeomorphism around 0 by the inverse function theorem.

There are $\tilde{\mathcal{U}}_0 \subseteq \tilde{\mathcal{U}} \times \mathbb{R}^{m-n}$ and $\tilde{V}_0 \subseteq \tilde{V}$ such that $\Phi|_{\tilde{\mathcal{U}}_0} : \tilde{\mathcal{U}}_0 \to \tilde{V}_0$ is a diffeomorphism

Then we have a diagram
$$\begin{tikzcd} \tilde{\mathcal{U}}_1 \arrow[r, "\tilde{F}"] & \tilde{V}_0 \\ \tilde{\mathcal{U}}_1 \arrow[u, "id"] \arrow[r, hook, "\text{canonical inclusion}"'] & \tilde{\mathcal{U}}_0 \subseteq \tilde{\mathcal{U}} \times \mathbb{R}^{m-n} \arrow[u, "\Phi|_{\tilde{\mathcal{U}}_0}"'] \end{tikzcd}$$
, where $0 \in \tilde{\mathcal{U}}_1 \subseteq \tilde{\mathcal{U}}$. $\tilde{\mathcal{U}}_1 \times \mathbb{R}^{m-n} \subseteq \tilde{\mathcal{U}} \times \mathbb{R}^{m-n} \supseteq \tilde{\mathcal{U}}_0$

Upon shrinking $(\mathcal{U}, \varphi)$ (to $\mathcal{U}_1 = \varphi^{-1}(\tilde{\mathcal{U}}_1)$) and changing (V, ψ) by $\Phi|_{\tilde{\mathcal{U}}_0}$ (by Φ^{-1}), we get the required charts. $\square$

Embedding = Immersion + homeo onto its image

Def A top space is *locally compact* $\forall x \in X, \forall$ open $\mathcal{U} \ni x, \exists$ open $V \ni x$ s.t. $\bar{V} \subseteq \mathcal{U}$ and $\bar{V}$ is compact.

Def A map is *proper* if the preimage of a compact set is compact.

Thm: A proper injective immersion is an embedding.

Pf: We have a lemma.

Lemma: Let X, Y be Hausdorff topological spaces, Y is locally compact and suppose $f : X \to Y$ is proper. Then f is closed. (maps closed to closed)

Assuming the lemma, since any manifold is locally compact, an injective proper immersion is a homeomorphism onto its image. $\square$

Pf of lemma: Let $A \subseteq X$ be closed. WTS $f(A)$ is closed, i.e. $Y \setminus f(A)$ is open, i.e. for any $y \in Y \setminus f(A)$ there is an open $\mathcal{U} \ni y$ such that $\mathcal{U} \cap f(A) = \emptyset$.

Let V be an open nbhd ? such that $\bar{V}$ is compact. Since f is proper, $f^{-1}(\bar{V})$ is compact in X, so the same goes for $C = f^{-1}(\bar{V}) \cap A$. Then $f(C)$ is compact, hence closed in Y.

Hence $\mathcal{U} = V \setminus f(C)$ is open in Y, $y \in \mathcal{U}$ and $\mathcal{U} \cap f(A) = \emptyset$. $\square$

Cor Any injective immersion between compact manifolds is an embedding.

Thm (structure thm for submersions)

(For open sets in Euclidean spaces, this is the implicit function theorem)

Let $F: X \to Y$ be a smooth map such that dF_x is surjective at $x \in X$.

Then there are charts (U, φ) and (V, ψ) such that

$$\begin{array}{ccc} U & \xrightarrow{F} & V \\ \varphi \downarrow & & \downarrow \psi \\ \tilde{U} & \xrightarrow{\tilde{F}} & \tilde{V} \end{array}$$

in the diagram $\tilde{F} = \pi|_{\tilde{U}}$ where $\pi: \mathbb{R}^n \to \mathbb{R}^m$

$(x_1, \dots, x_n) \mapsto (x_{n-m+1}, x_n)$

Pf: Next time

Def: Let X be a smooth manifold, Z a subspace of X.

Then we say that Z is a <u>submanifold</u> of X if $\forall z \in Z$ there is a chart (U, φ) such that $\varphi(U \cap Z) = \tilde{U} \cap \mathbb{R}^m \subseteq \mathbb{R}^n$, where $m \leq n$ and n is the dim X.

Lec 6 - 2/1 Implicit function thm, level sets and submanifolds

Thm Let $F: \mathcal{U} \to V$ be a smooth map such that

$$\mathcal{U} \subseteq \mathbb{R}^n, \quad V \subseteq \mathbb{R}^m$$

for some $x \in \mathcal{U}$, $dF_x : \mathbb{R}^n \to \mathbb{R}^m$ is surjective. Then, there is an open $x \in \mathcal{U}_0 \subseteq \mathcal{U}$ and a diffeomorphism $\varphi : \mathcal{U}_0 \to \varphi(\mathcal{U}_0)$ such that we have

$$\begin{tikzcd} \varphi(\mathcal{U}_0) \arrow[r, "\varphi^{-1}"] \arrow[dr, "\pi"'] & \mathcal{U}_0 \arrow[d, "F"] \\ & V \end{tikzcd}$$

projection onto last m coordinates

<u>Proof</u>: By assumption $dF_{x_0} = \left(\frac{\partial F}{\partial x_i}\Big|_{x_0} \right)$ ($1 \leq i \leq n$) is of rank m $\Rightarrow$ after possibly reshuffling the x_i's, we may assume the last m-columns span $\mathbb{R}^m$. Denote this $m \times m$ matrix by $M = \left(\frac{\partial F}{\partial x_j}\Big|_{x_0} \right)_{n-m+1 \leq j \leq n}$.

Then we can define $G : \mathcal{U} \to \mathbb{R}^n$

$$x = (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{n-m}, F(x))$$

By def, $dG_{x_0} = \begin{pmatrix} I & 0 \\ * & M \end{pmatrix} \Rightarrow \det(dG_{x_0}) = \det M \neq 0 \Rightarrow dG_{x_0}$ is an isomorphism, hence by the inverse function theorem, G is a local diffeomorphism.

$\Rightarrow \exists\, \mathcal{U}_0 \subseteq \mathcal{U}$ such that $G|_{\mathcal{U}_0} : \mathcal{U}_0 \to G(\mathcal{U}_0)$ is a diffeomorphism.

Moreover,

$$\begin{tikzcd} \mathcal{U}_0 \arrow[r, "G|_{\mathcal{U}_0}"] \arrow[dr, "F"'] & G|_{\mathcal{U}_0}(\mathcal{U}_0) \arrow[d, "\pi"] \\ & \mathbb{R}^m \end{tikzcd}$$

We are done by setting $\varphi = G|_{\mathcal{U}_0}$. $\square$

<u>Def</u> Let $F : X \to Y$ be a smooth map.

- We say that $x \in X$ is a <u>regular point</u> for F if dF_x is surjective $T_x X \to T_{F(x)} Y$
- We say that $y \in Y$ is a <u>regular value</u> for F if all points in $F^{-1}(y)$ are regular.

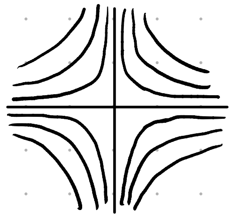
<u>Thm</u> Let $F : X \to Y$ be a smooth map between manifolds. Assume $x_0 \in X$ is a regular point. Then there are charts around x_0 and $y_0 = F(x_0)$: $(\mathcal{U}, \varphi)$ and (V, ψ) such that

$$\begin{tikzcd} \mathcal{U} \arrow[r, "F"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ \tilde{\mathcal{U}} \arrow[r, "\pi"] & \tilde{V} \end{tikzcd}$$

where π is the canonical projection.

<u>Pf</u>: Sim to immersion structure thm.

Examples (of submanifolds)



$dF_p = \begin{pmatrix} y_0 \\ x_0 \end{pmatrix}$ should be $(y_0\ x_0)$

$\Rightarrow$ If $x_0, y_0 \neq 0$, $F^{-1}(x_0 y_0)$ is a submanifold

Let $F: \mathbb{R}^2 \longrightarrow \mathbb{R}$
$(x,y) \longmapsto xy$

Then for a fixed $\alpha \in \mathbb{R}$, $F^{-1}(\alpha) = \{xy = \alpha\} \subseteq \mathbb{R}^2$

For any $\alpha \neq 0$, $F^{-1}(\alpha)$ is a submanifold.

Let's delete the non-positive x-plane, i.e, take $\mathcal{U} = \mathbb{R}^2 \cap \{x > 0\}$

Then define a map $\varphi: (x,y) \longmapsto (x, y - \frac{\alpha}{x})$, $\mathcal{U} \longrightarrow \mathcal{U}$. This is a diffeomorphism.

$\varphi(\mathcal{U} \cap F^{-1}(\alpha)) = \mathcal{U} \cap \mathbb{R}$, where $\mathbb{R} \subseteq \mathbb{R}^2$ is $\{(x,0), x \in \mathbb{R}\}$

$$\left(\begin{array}{l} (x_0, y_0) \in \mathcal{U} \cap F^{-1}(\alpha) \\ \Rightarrow \left.\begin{array}{l} x_0 > 0 \\ x_0 y_0 = \alpha \end{array}\right\} y_0 - \frac{\alpha}{x_0} = 0 \end{array} \right)$$

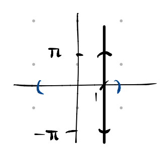
$\underline{S^1 \subseteq \mathbb{R}^2}$: $F(x,y) = x^2 + y^2$ $\mathbb{R}^2 \to \mathbb{R}$, $S^1 = F^{-1}(1)$

Take $\mathcal{U}_1 = \mathbb{R}^2 \setminus \{(x,0): x \leq 0\}$, $\tilde{\mathcal{U}}_1 = \mathbb{R}_{>0} \times (-\pi, \pi) \subseteq \mathbb{R}^2$.

$\varphi^{-1}: \tilde{\mathcal{U}}_1 \longrightarrow \mathcal{U}_1$
$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$

This is a diffeomorphism and we get a chart $(\mathcal{U}_1, \varphi)$ on $\mathbb{R}^2$

$\varphi(\mathcal{U}_1 \cap S^1) = 1 \times (-\pi, \pi) = \{(1, \theta) : \theta \in (-\pi, \pi)\} = (1 \times \mathbb{R}) \cap \tilde{\mathcal{U}}_1$



Want ( )
$\Rightarrow$ translate + rotate chart

__Thm__ Let $F: X \to Y$ be a smooth map and let $y \in Y$ be a regular value.

Then $F^{-1}(y) \subseteq X$ is a submanifold of dimension $\dim X - \dim Y$.
codimension $\dim Y$.

__Pf__ Let $Z = F^{-1}(y)$ and let $x \in Z$.

Since x is regular, there are charts on X and Y such that by implicit function thm

F becomes the canonical projection:

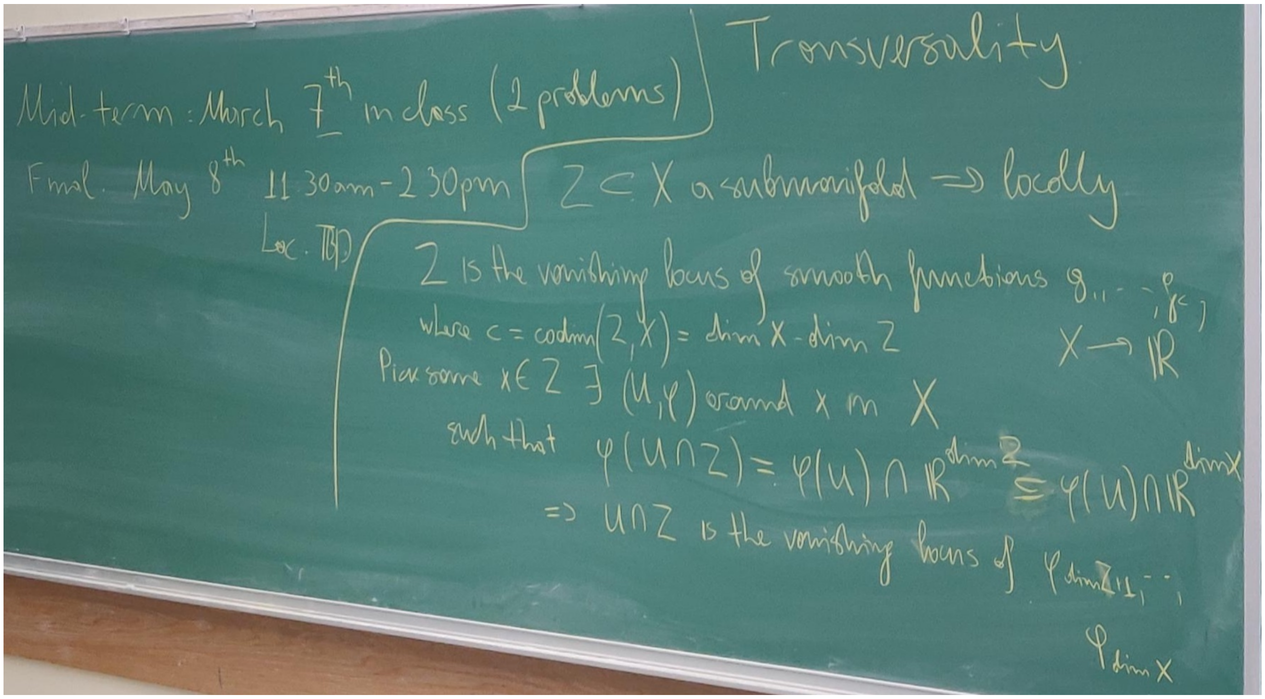
$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{F} & V \\ \varphi \downarrow & & \downarrow \psi \\ \tilde{\mathcal{U}} & \xrightarrow{\pi} & \tilde{V} \end{array}$$

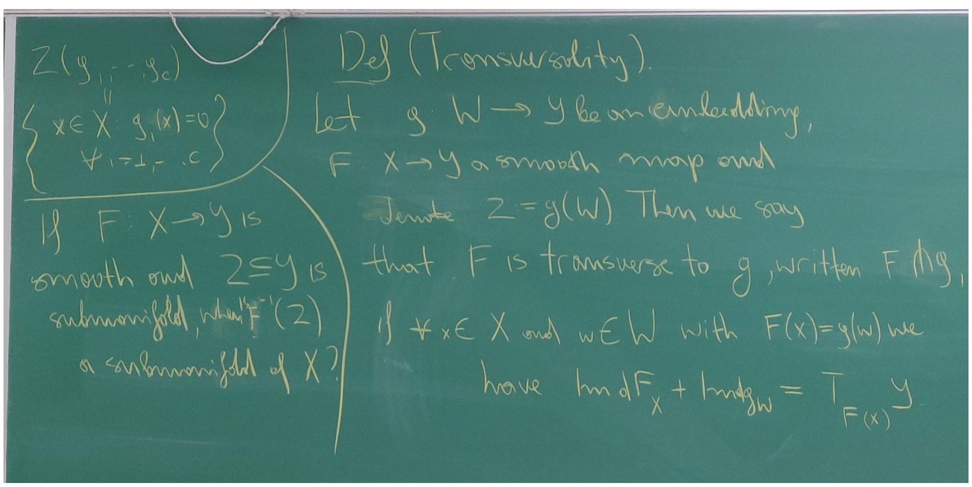
Then we can look at

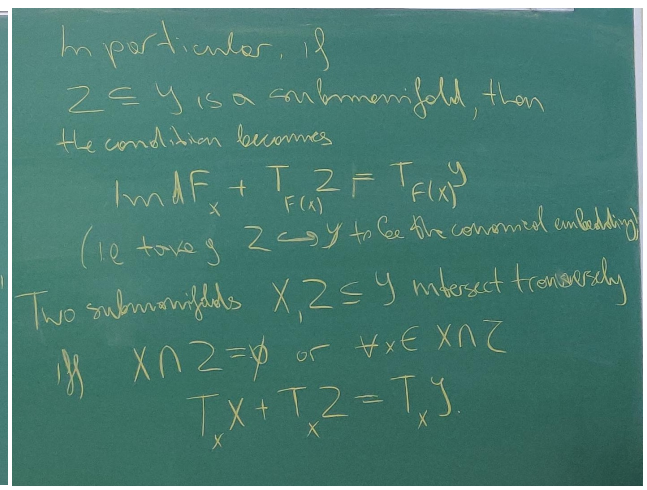
$\varphi(\mathcal{U} \cap Z) = \varphi(\mathcal{U} \cap F^{-1}(y)) = \varphi(\mathcal{U}) \cap \pi^{-1}(\psi(y))$

and $\pi^{-1}(\psi(y))$ is an affine space. $\pi^{-1}(\psi(y)) = \mathbb{R}^{\dim X - \dim Y} \times \{\psi(y)\}$ $\square$

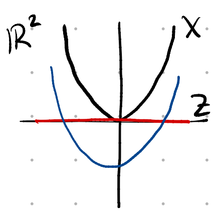
Lec 7 - 2/6





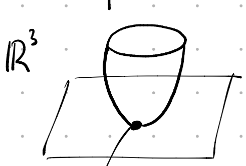


Some examples



Then $X \cap Z = \{0\}$ is not transverse since $T_0 X = T_0 Z$

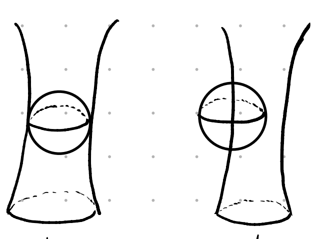
transverse



not transverse

$2 + 2 - 3 = 1$

$\dim X + \dim Z - \dim Y$



not transverse

transverse

Thm Let $F: X \to Y$ be smooth and $Z \subseteq Y$ be a submanifold such that $F \pitchfork Z$.

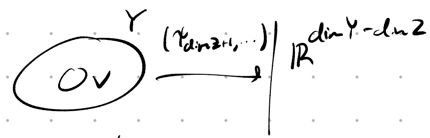
Then $F^{-1}(Z)$ is a submanifold of X of dimension $\dim X + \dim Z - \dim Y$,

i.e., $\operatorname{codim}(Z, Y) = \operatorname{codim}(F^{-1}(Z), X)$.

Pf Reduce to the case of regular values: let $x \in F^{-1}(Z)$ and a chart (V, ψ) on Y around $F(x)$ which is compatible with Z. Then $\psi(V \cap Z) = \psi(V) \cap \mathbb{R}^{\dim Z}$, $\mathbb{R}^{\dim Y} = \mathbb{R}^{\dim Z} \times \mathbb{R}^{\dim Y - \dim Z} \xrightarrow[\pi]{} \mathbb{R}^{\dim Y - \dim Z}$

Hence the restriction of F to $F^{-1}(V)$ is transverse to Z iff* $\pi \circ \psi \circ F|_{F^{-1}(V)}$ has $\vec{0}$ a regular value.

Then we're done by the regular value theorem. □



$$T_{F(x)} Z + \operatorname{Im} dF_x = T_{F(x)} Y$$

chart $\downarrow$

$$T_{F(x)}(Z \cap V) + \operatorname{Im} dF_x = T_{F(x)} Y$$

$\left(\pi \circ \psi \circ F|_{F^{-1}(V)}\right)^{-1}(0) = F^{-1}(V \cap Z) \longrightarrow$ sufficient, because being a submanifold is a local property.

Discussion

$\psi_1, \dots, \psi_c$ cutting out Z around $F(x)$

and transversality of F and Z then $\psi_1 \circ F, \dots, \psi_c \circ F$

are independent iff F is transverse to Z:

chain rule: $\boxed{d(\pi \circ \psi)\, dF = d(\pi \circ \psi \circ F)}$ *

$T_{F(x)} Z = \ker d(\pi \circ \psi)$, then apply thm:

$d(\pi \circ \psi) dF$ surj. iff $\operatorname{Im} dF + \ker d(\pi \circ \psi) = T_{F(x)} Z$

In the notation from lectures, the claim we had is:

Proposition. *The restriction of F to $F^{-1}(V)$ is transverse to Z iff 0 is a regular value for $\pi \circ \psi \circ F|_{F^{-1}(V)}$.*

Proof. The conditions in the proposition are linear, i.e. we're claiming something about the derivatives of these maps and they are linear maps.
Trasnversality means $T_{F(x)}Z + \operatorname{Im} dF_x = T_{F(x)}Y$. On the other hand, 0 is a regular value for $\pi \circ \psi \circ F|_{F^{-1}(V)}$ iff

$$d(\pi \circ \psi \circ F|_{F^{-1}(V)})_w = d(\pi \circ \psi)_{F(w)} \circ dF_w : T_w X \to \mathbf{R}^{\dim Y - \dim Z}$$

is surjective for all $w \in (\pi \circ \psi \circ F|_{F^{-1}(V)})^{-1}(0)$.
By our choice of chart, $V \cap Z$ is precisely the set where the map $\pi \circ \psi$ vanishes. Hence, its derivative $d(\pi \circ \psi)_z = 0$ on the subspace $T_z Z \subset T_z Y$ for all $z \in V \cap Z$. In fact, we have

$$T_z Z = \operatorname{Ker} d(\pi \circ \psi)_z.$$

The derivative is also a surjection

$$T_y Y \simeq \mathbf{R}^{\dim Y} \to \mathbf{R}^{\dim Y - \dim Z},$$

where $y \in V$. Hence, we're reduced to the following lemma which is pure linear algebra (and an exercise):

Lemma. *Let $L_1 : V_1 \to V_2$ and $L_2 : V_2 \to V_3$ be two linear maps such that L_2 is surjective. Then $L_2 \circ L_1$ is surjective if and only if $\operatorname{Ker} L_2 + \operatorname{Im} L_1 = V_2$.*

□

Pf: ($\Leftarrow$) Let $w \in V_3$. $\exists\, u$ s.t. $L_2 u = w$.

By hyp, $u = u^\circ + L_1 v$, where $L_2 u^\circ = 0$

$\Rightarrow L_2 u = L_2(u^\circ + L_1 v) = L_2 L_1 v$

($\Rightarrow$) Let $u \in V_2$. $\exists\, v$ s.t. $L_2 u = L_2 L_1 v$

$\Rightarrow u - L_1 v \in \ker L_2$,

so $u = u - L_1 v + L_1 v$ □

Lec 8 - 2/8

Homotopy:

Let $f, g : X \to Y$ are maps between topological spaces, we define

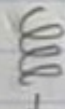
<u>Def</u>: A <u>homotopy</u> between f and g is a continuous map $H: X \times I \to Y$ such that $H(x,0) = f$, $H(x,1) = g$ and $I = [0,1]$

<u>Examples</u>:  $B =$  $H: B \times I \to B$

$(x,t) \mapsto tx + (1-t)g$

- $[-1,1] \to [-1,1]$: $x \mapsto x$ not isotopic
  $x \mapsto -x$
- Any two curves in $\mathbb{R}^n$ are homotopic.

<u>Def</u>: Two spaces X and Y are homotopic if there are $f: X \to Y$ and $g: Y \to X$ s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$.

$\mathbb{R} =$ 

$S^1 =$ 

Any $\gamma: [0,1] \to S'$ is $\simeq$ to $t \mapsto \exp(tn)$

<u>Def</u>: Two smooth maps $f, g: X \to Y$ between manifolds X and Y are <u>smoothly homotopic</u> if there is a smooth $H: X \times I \to Y$ s.t. $H(x,0) = f$ and $H(x,1) = g$

extends to a nbhd in $X \times \mathbb{R}$

<u>Def</u>: Let P be a property of a smooth map $f: X \to Y$. We say P is <u>stable under smooth homotopy</u> if $\forall H: X \times I \to Y$ s.t. $H(x,0) = f$, $\exists \varepsilon > 0$ s.t. $\forall t \in [0, \varepsilon)$ $H(x,t): X \to Y$ satisfies P

- Non-transverse intersections are not stable.
- Being nonconstant is stable.

Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, $H: \mathbb{R}^n \times I \to \mathbb{R}^m$ be a homotopy of linear maps. I.e. $\forall t \in I$, $H(-,t): \mathbb{R}^n \to \mathbb{R}^m$.
I.e. $H: I \to \mathrm{Mat}_{m\times n}$ is continuous.

lower semi continuous

Prop: Suppose given H as above and $\mathrm{rk}\, L \geq r$ for some r. ✓
Then, $\exists \varepsilon > 0$ s.t. $\forall t \in [0, \varepsilon)$, $\mathrm{rk}\, H(t) \geq r$. (rank can only have upper discontinuity)

Proof: Since $\mathrm{rk}\, L \geq r$, $\exists$ an $r\times r$-minor of L which has non-zero determinant. We may suppose it is top right corner.
Then, let $A: \mathrm{Mat}_{m\times n} \to \mathrm{Mat}_{r\times r}$
$M \longmapsto$ upper right $r\times r$ matrix

Then, we have a diagram:

$$\begin{tikzcd} I \arrow[r, "H"] \arrow[dr, "G"'] & \mathrm{Mat}_{m\times n} \arrow[d, "\det(A)"] \\ & \mathbb{R} \end{tikzcd}$$

G is continuous and $G(0) \neq 0$
$\Rightarrow \exists \varepsilon > 0\ \forall t \in [0, \varepsilon)$, $G(t) \neq 0$. □

Thm: Let $f: X \to Y$ be smooth and assume X is compact. Then the following properties are stable:

1. Immersion
2. Submersion
3. Local diffeomorphism
4. $f \pitchfork Z$
5. Embedding
6. Diffeomorphism

immersion that is a homeomorphism onto its image

Pf: 1, 2, 3, 4 in one go:

Let $n = \dim X$, $m = \dim Y$. 1 is saying $\mathrm{rk}\, df_x \geq n$.
2 —"— $\mathrm{rk}\, df_x \geq m$.
Traversality: $T_{f(x)}Z + \mathrm{Im}\, df_x = T_{f(x)}Y$
3 —"— $\mathrm{rk}\, df_x \geq n = m$
$\to$ 4 —"— we have $T_xX \xrightarrow{df_x} T_{f(x)}Y \xrightarrow{\pi} T_{f(x)}Y / T_{f(x)}Z$ — set diff
$f \pitchfork Z \Leftrightarrow \pi \circ df_x$ being surjective.
$\Leftrightarrow \mathrm{rk}(\pi \circ df_x) \geq n$

Now let $H: X \times I \to Y$ be a smooth homotopy. Then, we have say $\mathrm{rk}\, df_x \geq r$ $\forall x \in X$. Since H is smooth, we see that given $(x,0) \in X \times I$, $\exists$ open $U_x \times [0, \varepsilon)$ in $X \times I$ such that $\mathrm{rk}\, df_t|_y \geq r$ $\forall (y,t) \in U_x \times [0,\varepsilon)$.
We do this $\forall x \Rightarrow$ get a cover of X by $\{U_x, x \in X\}$. Use compactness to get a finite subcover $U_{x_1}, U_{x_2}, \dots, U_{x_\ell} \to$ let $\varepsilon = \min_{1\leq i \leq \ell} \varepsilon_i$.
$\Rightarrow \mathrm{rk}\, df_t|_x \geq r$ $\forall (x,t) \in X \times [0, \varepsilon)$.

H at time t

H at time t

let $\varepsilon = \min_{1 \leq i \leq \ell} \varepsilon_i$

Now we do 5 and 6:

5. Need to show that injectivity is stable for embeddings.
Suppose not. Then there are sequences t_i, x_i, y_i s.t. $f_{t_i}(x_i) = f_{t_i}(y_i)$ with $t_i \to 0$ $\forall i$ $x \neq y_i$. Then we may suppose $x_i \to \bar{x}$, $y_i \to \bar{y}$ by compactness. Then take limits $\Rightarrow f(\bar{x}) = \lim f_{t_i}(x_i) = \lim f_{t_i}(y_i) = f(\bar{y})$
$\Rightarrow \bar{x} = \bar{y}$. Then, look at the map $F: (x,t) \mapsto (f_t(x), t)$.
This is an immersion at $t=0$ at $\bar{x}$. Since the Jacobian is
$\begin{pmatrix} \frac{\partial f_t}{\partial x} & \boxed{*} \\ 0 & 1 \end{pmatrix}$. Then f is locally injective around $\bar{x}$, but that's a contradiction since any neighborhood contains x_i and y_i for $i \gg 0$.

6. f is a diffeomorphism. We know f_t is going to be an embedding $\forall t$ small enough. We're done if we show that f is surjective.
Since $\forall t$ small enough, f_t is a submersion $\Rightarrow f_t$ is an open map.
Since X is compact $\Rightarrow f_t$ is a closed $\Rightarrow f_t(X)$ is both open and closed in Y $\forall t$ small enough.

Then assuming X is connected (we always can), we see $f(X) = Y$. $\square$

Lec 9 - 2/13 - Morse Theory I

Some Analysis

A subset $X \subseteq \mathbb{R}^m$ is said to have <u>measure zero</u> if $\forall \varepsilon > 0$,

$\exists$ open cubes U_i such that the U_i's cover X and $\sum_i \mathrm{vol}(U_i) < \varepsilon$.

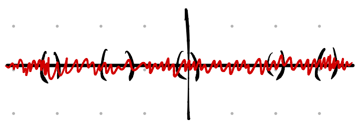
An <u>open cube</u> in $\mathbb{R}^n$ is a subset of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$.

Its volume is $\mathrm{vol} = \prod_{i=1}^{n} (b_i - a_i)$

<u>Examples:</u> 1) $\mathbb{Q} \subseteq \mathbb{R}$ is of measure zero.

2) More generally, any countable X is of measure zero.

3) $\mathbb{R} = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ has measure zero.

 use countable cover, thicken by diminishing thickness, e.g. $\frac{\varepsilon}{2^n}$

<u>Prop:</u> If $f: U \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ and $S \subseteq U$ of measure zero, then $f(S)$ also has measure zero

<u>Prop:</u> A countable union of sets of measure zero is measure zero.

<u>Pf</u>: $\sum_i \frac{1}{2^n} = 1$

<u>Def</u>: Let X be a smooth manifold. A subset $S \subseteq X$ has <u>measure zero</u> if for every chart (U, φ) on X, the set $\varphi(U \cap S)$ has measure zero.

<u>Thm</u> (Sard):

If $F: X \to Y$ is smooth, then the set of critical values of F, denoted C_F, has measure zero in Y.

<u>Remark</u>: We're not claiming anything about $F^{-1}(C_F)$; e.g. let $c: X \to Y$ be a constant map, $\dim X$, $\dim Y > 0$. Then every $x \in X$ is a critical pt for c.

<u>Cor</u>: If $Z \subseteq X$ is a submanifold with $\dim Z < \dim X$, then Z has measure zero in X.

Pf: $\iota: Z \to X$, then for any $z \in Z$, $d\iota_z: T_z Z \to T_z X$ is not surjective since $\dim Z < \dim X$, so z is a critical value. Hence any $z \in Z$ is a critical value and there's no others, so C_ι has measure zero.

If $f: X \to \mathbb{R}$ is a smooth function with X compact, then it has at least 1 critical pt.

Def: The Hessian of a function $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ is the matrix $H_f = \left(\frac{\partial^2 f}{\partial x_i x_j}\right)$.

Def: A critical point $x \in U$ for $f: U \to \mathbb{R}$ is non-degenerate if $\det H_{f_x} \neq 0$.

If X is a manifold, then we can define non-deg critical points for a smooth function using charts. By the chain rule, this is independent of charts.

Def: A smooth function $f: X \to \mathbb{R}$ is Morse if all critical points are non-degenerate.

Thm: Any manifold has a Morse function.

Thm: The set Morse functions on a compact manifold is dense and open in the space of smooth functions.

Thm: If $X \subseteq \mathbb{R}^N$, then for any smooth $f: X \to \mathbb{R}$, for almost all linear maps $L: \mathbb{R}^N \to \mathbb{R}$, $f + L$ is Morse.

Notation: D^n is the closed ball in $\mathbb{R}^n$ and $\partial D^n = S^{n-1}$

Def: An open n-cell is an open ball $D^n \setminus \partial D^n$.

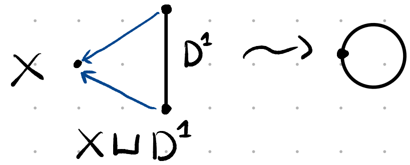
An open n-cell in a topological space X is an open U homeomorphic to an open n-cell

A cell decomposition of a top. space X is a disjoint union $X = \bigsqcup_{i \in I} e_i$, where I is an index set and each e_i is an n-cell for some n.

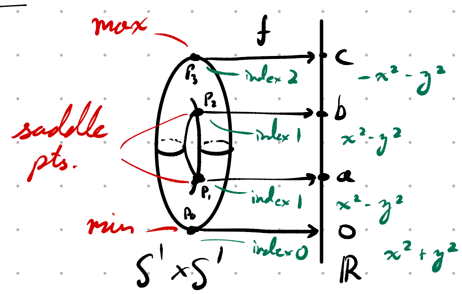
Ex $S^n = S^n \setminus \{p\} \sqcup \{p\}$

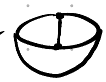
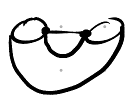
Def: If $f_\partial : S^{n-1} \to X$ is continuous, then we can attach an n-cell about f to X as follows: $X \cup_{f_\partial} D^n := X \sqcup D^n / \sim$, where $\sim$ is defined as $x \sim y$ for $x \in X$ and $y \in S^{n-1}$ is $x = f(y)$.

$f : S^{n-1} \to \{pt\} = X$



Example



- $h < 0$: $f^{-1}((-\infty, h]) = \emptyset$
- $h = 0$: $f^{-1}((-\infty, 0]) = \{p_0\}$
- $0 < h < a$: $f^{-1}((-\infty, h]) =$  $\overset{\text{homotopy}}{\sim} \{p_0\}$
- $a < h < b$: $f^{-1}((-\infty, h]) =$  $\sim$  attached 1 cell to the previous preimage
- $b < h < c$: $f^{-1}((-\infty, h]) =$  $\sim$  —— " ——
- $c \le h$: $f^{-1}((-\infty, h]) =$  $\sim$  —"—— 2 cell ——" ——

The index of a Morse function is the dimension of the space of negative eigenvalues of H_f.

Lec 10 - 2/15 Wrap up Morse, embedding thms into Euclidean spaces.

Morse lemma: Let $f: X \to \mathbb{R}$ be a Morse function and $x_0 \in X$ a critical point.

Then there are coordinates such that $\tilde{f}(x_1, \dots, x_n) = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2$

Def In the above notation, λ is the index of x_0.

Def Let $Y \subseteq X$ be a subspace of a top space.

We say that Y is a deformation retract of X if there is a homotopy

$H: X \times I \to X$ such that $H(x,0) = x \; \forall x \in X$

$$H(x,1) \in Y \text{ and } H(y,1) = y \; \forall y.$$

Thm (1st fundamental thm)

Let $f: X \to \mathbb{R}$ be Morse. Let $a, b \in \mathbb{R}$, $a < b$. Assume $f^{-1}([a,b])$ contains no critical pts. Then $f^{-1}((-\infty, a])$ is a deformation retract of $f^{-1}((-\infty, b])$

Thm (Reeb)

compact

Let $f: X \to$ be a Morse function with only two critical points.

Then X is homeomorphic to $S^{\dim X}$.

Thm (2nd fundamental thm)

Let $f: X \to \mathbb{R}$ be a Morse function and $c \in \mathbb{R}$ a critical value, $f(x_0) = c$.

Suppose the index of x_0 is λ. Then let $\varepsilon > 0$ be such that $f^{-1}([c-\varepsilon, c+\varepsilon])$ contains no other critical points. Then $f^{-1}((-\infty, c+\varepsilon])$ is obtained attaching a λ-cell to $f^{-1}((-\infty, c-\varepsilon])$.
(up to homotopy)

Def Let X be a smooth manifold. If $\{U_i\}$ is an open cover of X, we say that a family of smooth, non-negative functions $\rho_i: X \to \mathbb{R}$ is a partition of unity subordinate to $\{U_i\}$ if:

1) $\mathrm{supp}(\rho_i) \subseteq U_i$

2) $\forall x \in X$, only finitely many $\rho_i(x) \neq 0$

3) $\sum_i \rho_i = 1$

Thm: Any <u>compact</u> smooth manifold admits an injective immersion (= embedding) into some $\mathbb{R}^N$.

Let (U_i, φ_i) be a finite cover of X by charts and let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$.

"k"

Pf: We can define a map $F: X \to \mathbb{R}^{N = k(\dim X + 1)}$

$$x \mapsto (\underbrace{\rho_1(x)\varphi_1(x)}_{\text{each is a } \dim X\text{-tuple}}, \dots, \rho_k(x)\varphi_k(x), \rho_1(x), \dots, \rho_k(x))$$

Then F is an injective immersion.

Injectivity: if $F(x) = F(y) \Rightarrow \rho_i(x) = \rho_i(y)$ and wlog $\rho_1(x) \neq 0$,

then since $(\rho_1\varphi_1)(x) = (\rho_1\varphi_1)(y) \Rightarrow \varphi_1(x) = \varphi_1(y) \Rightarrow x = y$.

Immersion: If $dF_x(v) = 0$, then $(d\rho_i)_x(v) = 0 \ \forall i$, and by the product rule

$$\underbrace{(d\rho_i)_x(v)}_{0}\varphi_i(x) + \rho_i(x)(d\varphi_i)_x(v) = 0 \ \forall i \Rightarrow \rho_i(x)(d\varphi_i)_x(v) = 0$$

$\Rightarrow (d\varphi_i)_x(v) = 0$ since $\rho_i(x) \neq 0$ for some i $\Rightarrow v = 0$ by φ diffeomorphism.

Thm: Any n-dimensional, compact manifold embeds into $\mathbb{R}^{2n+1}$ and immerses into $\mathbb{R}^{2n}$.

Pf: Let $F: X \to \mathbb{R}^N$ be an embedding. We are claiming that for almost all

$\circ *$ $[v] \in \mathbb{RP}^{N-1}$, the composition $F_{[v]} = \pi_{[v]} \circ F : X \to \mathbb{R}^{N-1}$ is an embedding if $N > 2n+1$,

where $\pi_{[v]}: \mathbb{R}^N \to \mathbb{R}^{N-1}$ is the orthogonal projection.

Injectivity

If $F_{[v]}(x) = F_{[v]}(y)$

$\Leftrightarrow \pi_{[v]}(F(x) - F(y)) = 0$, i.e.,

$[F(x) - F(y)] = [v]$ in $\mathbb{RP}^{N-1}$.

By Sard, we see the set of such $[v]$ is of measure zero.

Immersion Let $TX = \coprod_{x \in X} T_xX = \{(x,v) : v \in T_xX\}$.

Then TX is a $2n$-dim manifold and it has a natural projection $p: TX \to X$.

$$(x,v) \mapsto x$$

Then $dF_{[v]}: TX \to T\mathbb{R}^{N-1} = \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$.

If $(dF_{[v]})_x(w) = 0$, then $\pi_{[v]}(dF)_x(w) = 0$ $\quad TX \setminus (x,0) \to \mathbb{RP}^{N-1}$, $(x,w) \mapsto [dF_x(w)]$

$\Rightarrow [(dF_x)(w)] = [v]$ in $\mathbb{RP}^{N-1}$ hence by Sard the set of such $[v]$ has measure zero since $2n < N-1$.

↳ Reduce TX to $SX \Rightarrow \dim: 2n \to 2n-1 \Rightarrow 2n-1 < N-1$

Lec 11 - 2/20 Embedding results in the noncompact case.

Continuing from last time: (Clarifying)

$F_{[v]} = \pi_{[v]} \circ F$: looking for $[v] \in \mathbb{RP}^{N-1}$ s.t. $F_{[v]}(x) = F_{[v]}(y)$ for some $x, y \in X$, $x \neq y$.

$$\Rightarrow \pi_{[v]}(F(x) - F(y)) = 0 \Leftrightarrow F(x) - F(y) \text{ is parallel to } v, \text{ i.e.}$$

$$[t(F(x) - F(y))] = [v]$$

diagonal

This is expressable as the map
$$X \times X / \Delta \longrightarrow \mathbb{RP}^{N-1}$$
$$(x, y) \longmapsto [F(x) - F(y)]$$

$\dim(X \times X/\Delta) = 2n$. $\dim \mathbb{RP}^{N-1} = N-1$. By assumption, $2n < N-1$

$\Rightarrow$ map is not surjective for almost choices of $[v]$ by Sard's Theorem, i.e.,

choosing $[v]$ in the complement.

Non-compact case: We need to show that there is an injective immersion into some $\mathbb{R}^N$ for a start.

Def: An exhaustion function on a manifold X is a smooth $f : X \to \mathbb{R}$ such that

for all $a \in \mathbb{R}$, the preimage $f^{-1}((-\infty, a])$ is compact.

Such an f is proper.

Prop: Any manifold X admit such an exhaustion function.

Pf: Let $\{U_i\}_{i \in \mathbb{N}}$ be an open cover of X and let

$\{V_j\}$ be a (locally-finite) refinement of $\{U_i\}$ such that $\bar{V}_j$ are compact. $(\bar{V}_{\alpha(i)} \subseteq U_i)$

Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$ and so that $\rho_i|_{\bar{V}_{\alpha(i)}} = 1$.

Take a sequence $a_i \in \mathbb{N}$, $\lim a_i = \infty$. Set $f := \sum_{i \geq 1} a_i \rho_i$

Fix $a \in \mathbb{R}$. Then there is $n \in \mathbb{N}$ s.t. $a_i > a \ \forall i > n$.

We claim $f^{-1}((-\infty, a]) = f^{-1}([0, a]) \subseteq \bigcup_{i=1}^{n} \bar{V}_{\alpha(i)}$

This is because for $i_0 > n$, $f(x) = \sum_{i \geq 1} a_i \rho_i(x) = \sum_{i \neq i_0} a_i \rho_i(x) + a_{i_0} > a$ for $x \in \bar{V}_{\alpha(i_0)}$.

Thm: Any X admits an injective immersion into some $\mathbb{R}^N$.

Pf: Let f be a non-negative exhaustion function on X.

Let $\{(U_i, \varphi_i)\}$ be an open cover by charts, indexed by $\mathbb{N}$.

Define $Y_i := f^{-1}([i, i+1])$. Let $\varepsilon > 0$ be small enough (e.g. $\varepsilon = \frac{1}{10}$)

and define $X_i := f^{-1}((i-\varepsilon, i+1+\varepsilon)) \cap (U_1 \cap \cdots \cap U_{k_i})$ where $U_1 \cup \cdots \cup U_{k_i} \supseteq Y_i$

So X_i is an open neighborhood of Y_i. It is also covered by finitely many charts.

$N = 2\dim X_i + 1$ → use proj. to reduce.

Hence we get an injective immersion ψ_i into Euclidean space for X_i.

For each i, take a smooth non-neg. function $\sigma_i : X \to \mathbb{R}$ such that $\sigma_i|_{Y_i} = 1$, supp $\sigma_i \subseteq X_i$.

bump function

Then the map $F : X \to \mathbb{R}^{2N+1}$

$$x \mapsto \Big(\sum_{\text{even } i} \sigma_i(x)\psi_i(x), \sum_{\text{odd } i} \sigma_i(x)\psi_i(x), f(x)\Big) \to \text{consult pf from last lecture}$$

is an injective immersion. □

So if we have an injective immersion, by the projection construction, we get an injective immersion $F : X \to \mathbb{R}^{2\dim X + 1}$. It needn't be proper.

We may assume $|F(x)| \leq 1$ $\forall x \in X$ (by taking a diffeo, e.g. $x \mapsto \frac{x}{1+|x|}$)

Take an exhaustion function $f : X \to \mathbb{R}$ (non-negative) and define a new map

$G : X \to \mathbb{R}^{2\dim X + 2}$ Then reduce the dimension of $\mathbb{R}^{2\dim X + 2}$ by 1 using

$x \mapsto (F(x), f(x))$ the projection along some $v \in \mathbb{R}P^{2\dim X + 1}$. We may suppose $|v| = 1$

and the last component of v is not ± 1.

Then $G_{[v]} := \pi_{[v]} \circ G : X \to \mathbb{R}^{2\dim X + 1}$ is a <u>proper</u> injective immersion, so an embedding. Pf: Let $K \subseteq \mathbb{R}^{2\dim X + 1}$ be compact. Then choose an $a > 0$ so that $K \subseteq \{x : |x_{2\dim X+1}| \leq a\}$. Want to show $G_{[v]}^{-1}(K)$ is compact. Since $\pi_{[v]}(x) = x - (x, v)v$

$$\Rightarrow G_{[v]}(x) = G(x) - \langle G(x), v\rangle v = (F(x), f(x)) - \langle (F(x), f(x)), (v', v_{2\dim X+1})\rangle (v', v_{2\dim X+1}),$$

where $v = (v', v_{2\dim X+1})$, $\quad = (F(x), f(x)) - (F(x)\cdot v' + f(x)\, v_{2\dim X+1})(v', v_{2\dim X+1})$

So, $G_{[v]}(x) = (*, f(x)(1 - v_{2\dim X+1}^2) - (F(x)\cdot v')\, v_{2\dim X+1})$ $\exists A > 0$ such that $\quad A = \dfrac{a+1}{|1 - v_{2\dim X+1}^2|}$

$G_{[v]}^{-1}(K) \subseteq f^{-1}([-A, A])$, which shows that $G_{[v]}^{-1}(K)$ is compact.

Lec 12 - 2/22 Manifolds with boundary

Clarifying proof from last time

$$G_{[v]} = \pi_{[v]} \circ G = G - \langle G, v\rangle v \qquad K \subseteq \{x \mid |x_{2\dim X+1}| \le a\}, \ K \text{ compact in } \mathbb{R}^{2\dim X+1}$$

Goal: $G_{[v]}^{-1}(K)$ compact. $x \in K$

$$\Rightarrow G_{[v]}(x) = (F(x), f(x)) - \langle (F(x), f(x)), \overbrace{(v', v_{2\dim X+1})}^{v} \rangle (v', v_{2\dim X+1})$$

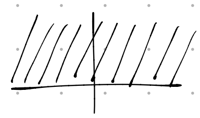
$$= \left(*, f(x)(1 - v_{2\dim X+1}^2) - (F(x)\cdot v')\, v_{2\dim X+1}\right)$$

By Δ-inequality, we get

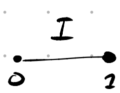
$$|f(x)|(1 - v_{2\dim X+1}^2) - |(F(x)\cdot v')\, v_{2\dim X+1}| \le a \quad \Rightarrow \quad |f(x)|(1 - v_{2\dim X+1}^2) - 1 \le a$$

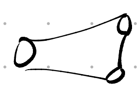
$$\Rightarrow |f(x)| \le \frac{a+1}{(1 - v_{2\dim X+1}^2)} =: A \ \Rightarrow G_{[v]}^{-1}(K) \subseteq f^{-1}([-A, A]) \longrightarrow \text{compact.} \ \square$$

Manifolds with boundary

$H_n = \{x \in \mathbb{R}^n : x_n \ge 0\}$ - upper half-plane 

Def A <u>manifold with boundary</u> is a topological space X with an open cover by charts: $(\mathcal{U}_i, \varphi_i)$, where $\varphi_i : \mathcal{U}_i \to \varphi_i(\mathcal{U}_i) \subseteq H_n$ are homeomorphisms and transition functions are smooth.

Examples  

 - pair of pants

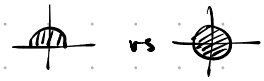
Prop/Def Let X be a manifold with boundary & $\dim X = n$.

Then $\mathrm{Int}(X)$ is an n-dim manifold (without boundary).

Let ∂X be the set of $x \in X$ s.t. $\exists$ a chart $(\varphi, \mathcal{U})$ with $x \in \mathcal{U}$ and $\varphi(x) \in \partial H_n$.

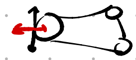
Then ∂X is well-defined and is a manifold of $\dim = n-1$.

Pf: ∂X is well-defined because if (V, ψ) is another chart and $\psi(x) \notin \partial H_n$, then we get a diffeomorphism between an open disk and a half-disk, which is a contradiction since the latter is not open in $\mathbb{R}^n$.



Aside: In fact, $\mathbb{R}^n$ and H_n are not homeomorphic, e.g. $\pi_n(\mathbb{R}^n \setminus pt) \not\cong 0$, while $\pi_n(H_n \setminus pt) \cong 0$.

Def The tangent space to a manifold with boundary is
the equivalence class of curves $\sigma: (-\varepsilon, 0] \to X$ or $\sigma: [0, \varepsilon) \to X$ s.t. $\sigma(0) = x$.
under $\sigma_1 \sim \sigma_2$ iff $\sigma_1'(1) = \sigma_2'(1)$

This is introduced to deal with normal vectors at the boundary 

Lemma Let X be a manifold and $f: X \to \mathbb{R}$ a smooth function
with regular value a. Then $Z = f^{-1}((-\infty, a])$ is a manifold with boundary $\partial Z = f^{-1}(a)$.

Pf Since a is regular, f looks like the canonical projection onto the last coordinate
$\pi: (x_1, \dots, x_n) \mapsto x_n$, $n = \dim X$

So around $x_0 \in f^{-1}(a)$, we have a chart $\varphi: \mathcal{U} \to \varphi(\mathcal{U})$, $\varphi(\mathcal{U} \cap Z) \cong \overset{\overset{(-\infty, a]}{\downarrow}}{H_n^-}$

Thm (Sard)
Let $f: X \to Y$ be a smooth map where X is a manifold with boundary
and Y is a manifold (so $\partial Y = \phi$).
Then the set of critical values for f, C_f has measure zero.

Pf: f defines 2 smooth maps. $f_0: \mathrm{Int}(X) \to Y$ and $\partial f := f|_{\partial X}: \partial X \to Y$.
Then both have sets of critical values of measure zero, so we get C_f is of measure zero
↳ usual Sard

Thm Let X be a manifold with boundary, Y a manifold ($\partial Y = \phi$) and $f: X \to Y$ smooth.
If $Z \subseteq Y$ is a submanifold and $f \pitchfork Z$, and $\partial f \pitchfork Z$, then $f^{-1}(Z)$ is a submanifold of X
with boundary $\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X = \partial f^{-1}(Z)$

Pf: The question is local, so we may assume $X \subseteq H_n$ and $Y = \mathbb{R}^m$.
If $x \in f^{-1}(Z)$ is not on the boundary of X, we are done by transversality in the case of manifolds without boundary.

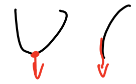
Suppose $x \in f^{-1}(Z) \cap \partial X$. Then we have an extension of, denoted $\tilde{f}: \mathcal{U} \to \mathbb{R}^m$ where is an open ball in $\mathbb{R}^n$ and $x \in \mathcal{U}$. Then, since $df_x = d\tilde{f}_x$, we see that $\tilde{f} \pitchfork Z$ locally.

By transversality, we get that $\tilde{f}^{-1}(Z)$ is a submanifold of $\mathbb{R}^m$,

and we claim that $\tilde{f}^{-1}(Z) \cap H_n = f^{-1}(Z)$.

Finish next time...

Tangent Space: Lec 13 Author: Persy

i.e. 

Def Tangent Space to a mfd w. boundary is the space of equivalence classes of curves $\gamma : [-\varepsilon, 0] \to X$ or $[0, \varepsilon) \to X$, $\gamma(0) = x$, under $\gamma_1 \sim \gamma_2$, $\gamma_1'(1) = \gamma_2'(1)$

Lemma:

Let X be a manifold and $f : X \to \mathbb{R}$ a smooth func with regular value $a \in \mathbb{R}$

Then $Z = f^{-1}((-\infty, a])$ is a manifold with boundary $\partial Z = f^{-1}(a)$

Pf.
Since a is regular, f looks like the canoical projection onto the last coordinate

$\pi : (x_1, \cdots, x_n) \to x_n$ so around a we have the chart $\varphi : U \to \varphi(U) = H_n$

Thm (Sard)

Let $f : X \to Y$ be a smooth map where X is a manifold with boundary and Y is a mfd (so $\partial Y = \phi$)
Then the set of crifical values for f C_f has measure 0.

Pf: f defines 2 smooth maps $f_0 : \text{Int } X \to Y$
$\partial f = f|_{\partial X}\ \partial X \to Y$

Then both sets of critical values of measure 0, so we get C_f measure 0
usual Sards

Thm: Let X be a manifold with boundary, Y a mfd $(\partial Y = \phi)$ and $f : X \to Y$ smooth if $Z \subseteq Y$ is a submfd and $f \pitchfork Z$ and $\partial f \pitchfork Z$. then $f^{-1}(Z)$ is a submanifold of X with boundary $\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X$.

Pf: local question. may assume $X \subseteq H_n$ and $Y = \mathbb{R}^n$ if $x \in f^{-1}(Z)$ is not on the boundary we've done by the prev. transversality.

<u>Suppose</u> $x \in f^{-1}(Z) \cap \partial X$.

Then we have an extension of f, denoted by $\tilde{f}: U \to \mathbb{R}^m$

Where U is open ball in $\mathbb{R}^n$ and $x \in U$.

Then since $df_x = d\tilde{f}_x$, we see that $\tilde{f} \pitchfork Z$ locally

by transversality, we get that $\tilde{f}^{-1}(Z)$ is a submfd of $\mathbb{R}^n$ and we

Claim $\underbrace{\tilde{f}^{-1}(Z)}_{=W} \cap H_n = f^{-1}(Z)$

Define $\pi_W: W \to \mathbb{R}$

$\quad\quad\quad\quad \cup$

$(x_1, \dots, x_n) \mapsto x_n$

By lemma: Since $\pi_W^{-1}([0,\infty)) = f^{-1}(Z)$ if 0 is a regular value for π_W, we are done.

Suppose 0 is not a regular value. Hence there is some $x \in \pi^{-1}(0)$ ~~st.~~ $\Rightarrow$ $d\pi_x = 0$ & $\pi(x) = 0$

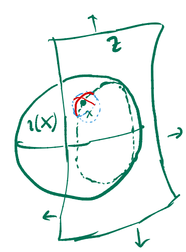
i.e. the last coordinate of x is zero.

By transversality: $\operatorname{codim}(T_xW, \underbrace{T_xX}_{\dim n}) = \operatorname{codim}(T_{f(x)}Z, T_{f(x)}Y)$ $\ast$

Note that $T_xW = (df_x)^{-1}(T_{f(x)}Z)$

By assumption on X. we see that $T_xW = (d\,\partial f_x)^{-1}(T_{f(x)}Z)$

Since $\partial f \pitchfork Z \Rightarrow \operatorname{Codim}(T_xW, \underbrace{T_x\partial X}_{\dim n-1}) = \operatorname{Codim}(T_{f(x)}Z, T_{f(x)}Y)$



Contradict $\ast$ Q.E.D

Classification of 1-manifold and application

<u>Thm</u> (Topological manifold)

1) Any non-compact connected 1-manifold w.o. boundary is homeomorphic to $\mathbb{R}$.

2) Any non-compact connected 1-mfd w. boundary is homeomorphic to $[0,\infty)$

3) Any compact connected 1-mfd w.o. boundary is homeomorphic to S^1

4) Any compact connected 1-mfd w. boundary is homeomorphic to $[0,1]$.

Thm

Any topological 1-mfd admits a unique smooth structure and therefore the classification thm holds in the category of smth mfd.

Cor Let X be a compact 1-mfd. w. boundary, then

$$\underbrace{H(\partial X)}_{\#\text{ pts in } \partial X} = 0 \pmod 2$$

$\#$ pts in ∂X. (if $S^1 \Rightarrow 0$ pts

interval $[0,1] \Rightarrow 2$ pts]

boundary 1   boundary 2

Cor Let X be a compact mfd w. boundary $i : \partial X \hookrightarrow X$

then i doesnot have a left inverse i.e. there is no r s.t.

$$r : X \to \partial X, \quad r \circ i = Id_{\partial X}$$

Pf Assume such r exists, then let $z \in \partial X$ be a regular value

Then if $n = \dim X$. we see that:

$$\operatorname{Codim}(r^{-1}(z), X) = \operatorname{Codim}(\underset{0}{z}, \underset{n-1}{\boxed{\partial X}}) = n-1.$$

$$\dim r^{-1}(z) = 1$$

$$\underbrace{\partial r^{-1}(\{z\})}_{} = \{z\} \quad \text{contradiction}$$

because of $r \circ i = Id_{\partial X}$ so there is only z here

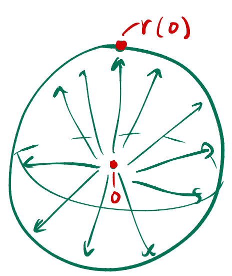
need even number of pts on boundary

one pt.

$$\partial(r^{-1}(z)) = r^{-1}(z) \cap \partial X$$

$$= \{z\} \quad \text{since } r \circ i = id_{\partial X}$$

if $z_1 \in \partial X$, $r(z_1) = z = r\,i(z_1) = z_1$

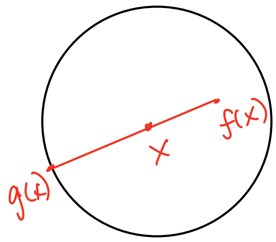


neither smooth nor continuous

Cor (W.B. fixed pt thm)

Let B^n be the closed ball in $\mathbb{R}^n$, any $f: B^n \to B^n$ has a fixed pt i.e. $\exists x \in B^n$ $f(x) = x$.

Pf Define $g: B^n \to S^{n-1}$ assume f has no fixed pt as follows for any $x \in B^n$



let $g(x) - f(x) = t(x - f(x))$ for some $t_x \geq 1$

Observe that by contradiction. $g|_{S^{n-1}} = \mathrm{Id}_{S^{n-1}}$

We WTS that g is smooth, t_x varys smoothly w. x.

Write

$g(x) = t_x x + (1 - t_x) f(x) = t_x(x - f(x)) + f(x)$

norm?

$1 = t_x^2 |x - f(x)|^2 + 2t_x (x - f(x)) \cdot f(x) + |f(x)|^2$

t_x is a positive soln to

$t^2 |x - f(x)|^2 + 2t_x(x - f(x)) \cdot f(x) + (|f(x)|^2 - 1)$

and hence a smooth func of x.

Thm (genericity of transversality)

Let X be a mfd w. boundary and let S and Y be mfd.

Suppose $F: X \times S \to Y$ be a smth map and Z be a submfd of Y and $F \pitchfork Z$, $\partial F \pitchfork Z$. then for almost $\forall s \in S$:

$$f_s := F(-, s) \pitchfork Z \text{ and } \partial f_s \pitchfork Z.$$

Pf. We're going to consider a pullback diagram:

$$\begin{tikzcd} W \arrow[r] \arrow[d, hook, "r"'] & Z \arrow[d, hook, "i"] \\ X \times S \arrow[r, "F"'] & Y \end{tikzcd}$$

then we're going to show that the following are equivalent:

1) $s \in S$ is a regular value of the projection $\pi_W : W \to S$.

2) $s \in S$. $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$

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Thm (Genericity of Transversality)

Let $F: \overset{\text{w/}\,\partial}{X} \times S \to Y \supseteq Z$, and $F \pitchfork Z$, $\partial F \pitchfork Z$

Then $f_s := F(-, s) \pitchfork Z$ and $\partial f_s \pitchfork Z$ for almost all $s \in S$.

Pf: Consider the diagram

$$\begin{tikzcd} W \arrow[r, "F"] \arrow[d, hook] \arrow[dd, bend right=60, "\pi_W"'] & Z \arrow[d, hook] \\ X \times S \arrow[r, "F"] \arrow[d, "\pi"] & Y \\ S & \end{tikzcd}$$

The claim is:

1) $f_s \pitchfork Z$ if s is a regular value of π_W.

2) $\partial f_s \pitchfork Z$ if s is a regular value of $\partial \pi_W$.

(Pf of $1 \Rightarrow 2$).

Pf of 1) Assume $s \in S$ is a regular value of π_W.

WTS $\operatorname{Im} d(f_s)_x + T_{F(x,s)} Z = T_{F(x,s)} Y$, where $x \in X$.

We know that $F \pitchfork Z \Rightarrow$ for any v in $T_{F(x,s)} Y$, there are

$(v_1, v_2) \in T_{(x,s)}(X \times S)$ and $w \in T_{F(x,s)} Z$ such that $dF_{(x,s)}(v_1, v_2) + w = v$

We'd like $v_2 = 0$, but it wouldn't be the case automatically.

Since s is regular, we get a vector $(v_1', v_2) \in T_{(x,s)} W$ such that $(d\pi_W)_{(x,s)}(v_1', v_2) = v_2$.

Then the vector $v_1 - v_1'$ is our desired solution,

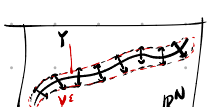
since $d(f_s)_x (v_1 - v_1') + w = dF_{(x,s)}(v_1, v_2) - dF_{(x,s)}(v_1', v_2) + w$

$\Rightarrow$ we can modify w by $dF_{(x,s)}(v_1', v_2) \in T_{F(x,s)} Z$ to get our $v \in T_{F(x,s)} Y$. $\square$

Given f and Z, how do we construct such deformations F?

If $f: X \to \mathbb{R}^N$, then we can simply take $F: X \times \mathbb{R}^N \to \mathbb{R}^N$
$$(x, s) \longmapsto f(x) + s$$

This is a submersion, so transverse to anything we like.



$Y^\varepsilon = \{ x \in \mathbb{R}^N : |x - y| < \varepsilon \text{ for some } y \in Y \}$.

Then get F by taking $F: X \times B_\varepsilon \longrightarrow Y^\epsilon \xrightarrow{\pi} Y$

$$(x, s) \longmapsto f(x) + s \mapsto y$$

Normal bundle

In general, let $Z \xhookrightarrow{\iota} X$ be a submanifold.

Then we have an exact sequence
$$\begin{tikzcd} & & & TX|_Z / T_Z \arrow[d, equal] & \\ 0 \arrow[r, "d\iota"] & TZ \arrow[r] & TX|_Z \arrow[r] & \mathrm{coker}(d\iota) \arrow[r] & 0 \end{tikzcd}$$

Def The <u>normal bundle</u> of the embedding $\iota : Z \hookrightarrow X$ is $N_{Z/X} := TX|_Z / TZ$.

$$\begin{tikzcd} TZ \arrow[r, hook] \arrow[d] & TX|_Z \arrow[dl] \\ Z & \end{tikzcd}$$

If $X \subseteq \mathbb{R}^k$, we have $T_x X \subseteq \mathbb{R}^k \; \forall x \in X$.

Then we define $N_{X/\mathbb{R}^k, x} := (T_x X)^\perp$ and their union $\bigsqcup_{x \in X} N_{X/\mathbb{R}^k, x}$ is called the normal bundle of X in $\mathbb{R}^k$.

<u>Prop</u> Let $X \subseteq \mathbb{R}^k$ be a submanifold. Then $N_{X/\mathbb{R}^k}$ is a submanifold of $X \times \mathbb{R}^k$ of dimension k and the canonical projection $\pi : N_{X/\mathbb{R}^k} \to X$ is a submersion.

<u>Pf</u> If A is a linear map, $A : \mathbb{R}^k \to \mathbb{R}^m$, then its transpose is A^t defined $\langle Av, w \rangle = \langle v, A^T w \rangle$.

$A^t : \mathbb{R}^m \to \mathbb{R}^k$ and $\mathrm{Im}(A^t) = (\ker(A))^\perp$ if A surjective. Moreover, if A surjective, $AA^t : \mathbb{R}^m \to \mathbb{R}^m$ is in $GL_m(\mathbb{R}^m)$, i.e. invertible.

Let $\mathcal{U} \subseteq \mathbb{R}^k$ be open and φ a submersion $\varphi : \mathcal{U} \to \mathbb{R}^m$ such that $\varphi^{-1}(0) = \mathcal{U} \cap X$.

Then set $N_{X/\mathbb{R}^k}(\mathcal{U}) = N_{X/\mathbb{R}^k} \cap (\mathcal{U} \times \mathbb{R}^k)$. Note that $T_x X = \ker(d\varphi_x) \; \forall x \in \mathcal{U} \cap X$.

Define two maps:
$$\Phi : \mathcal{U} \times \mathbb{R}^k \to \mathcal{U} \times \mathbb{R}^m \quad (x, v) \longmapsto (x, d\varphi_x(v))$$
$$\Psi : \mathcal{U} \times \mathbb{R}^m \to \mathcal{U} \times \mathbb{R}^k \quad (x, w) \longmapsto (x, (d\varphi_x)^t(w)).$$

Then, since $\begin{array}{c} x \longmapsto d\varphi_x \circ d\varphi_x^t \\ \mathcal{U} \cap X \longrightarrow GL_m(\mathbb{R}) \end{array}$ is smooth, we see that $\Phi \circ \Psi$ is a diffeomorphism.

So Ψ is a diffeomorphism onto its image $N_{X/\mathbb{R}^k}(\mathcal{U})$ and $\Psi^{-1} : N_{X/\mathbb{R}^k}(\mathcal{U}) \to \mathcal{U} \times \mathbb{R}^k$ is our chart.